Dominance Solvability in Random Games

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Abstract. We study the effectiveness of iterated elimination of strictly-dominated actions in random two-player games. We show that dominance solvability of games is vanishingly small as the number of at least one player's actions grows. Furthermore, conditional on dominance solvability, the number of iterations required to converge to Nash equilibrium grows rapidly as action sets grow. Nonetheless, at least when one of the players has a small action set, iterated elimination simplifies the game substantially by ruling out a sizable fraction of actions. This is no longer the case as both players' action sets expand. With more than two players, iterated elimination becomes even less potent in altering the game players need to consider. Technically, we illustrate the usefulness of recent combinatorial methods for the analysis of general games.

Keywords: Random Games, Dominance Solvability, Iterated Elimination

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1. Introduction

1.1. Overview. First introduced by Moulin (1979) in the context of voting, dominance solvability relies on a straightforward prescription. If a player has an action that generates worse payoffs than another regardless of what other players select—a strictly dominated action—she should never use it. When the structure of the game is commonly known, other players can infer their opponents' strictly dominated actions and assume they will not be played. With those strictly dominated actions eliminated, the resulting, reduced game may have further strictly dominated actions that can then be eliminated, and so on and so forth. This iterative procedure allows players to restrict the set of relevant actions they should consider. If it converges to a unique action profile, that profile constitutes a Nash equilibrium, and the game is dominance solvable.

Dominance solvable games are appealing on both simplicity and robustness grounds. Indeed, players do not have to hold precise beliefs about opponents or even assess at great accuracy the payoffs resulting from each action profile—whether or not a game is dominance solvable as well as the resulting predictions depend only on *ordinal* comparisons of players' payoffs. It is no surprise that dominance solvable games have found good use in a wide range of applications with scholars attempting to identify naturally-occurring dominant solvable games or implement desirable outcomes through protocols inducing dominant-solvability. Our goal here is to provide a full picture of the distribution of various outcomes resulting from iterated elimination of strictly dominated actions in large two-player random games.

We consider random games, where the ranking of payoffs resulting from all possible profile actions is determined uniformly at random.[1] This allows us to analyze the likelihood of different dominance features within games of varying sizes. We study three aspects that serve as proxies for game "simplicity." First, we look at the probability a game is dominance solvable. Second, conditional on a game being dominance solvable, we consider the number of iterations of strictly dominated actions necessary to reach the resulting Nash equilibrium. Last, whether or not a game is dominance solvable, we consider the size of the surviving set of actions for each player. To the extent that smaller actions sets simplify the task of a player, the volume of surviving actions provides some indication as to the game's complexity.

Perhaps confirming common wisdom, we show that the probability a game is dominance solvable vanishes quickly as any player's action set grows. For $2 \times n$ games, this probability

[1]Throughout, we consider dominance solvability restricting attention to pure strategies. We discuss extensions to mixed-strategy notions at the end of the paper.

is strictly decreasing in n and proportional to $n^{-1/2}$. As we increase the action sets of both players, for $m \times n$ games with $m \leq n$, the probability a game is dominance solvable is $n^{-\Theta(m)}$. These results suggest the rarity of dominance solvable games. Consequently, they indicate how special many of the games the literature focuses on are. They also make the classic virtual implementation results à la Abreu and Matsushima (1992a) appear even more remarkable than before: a good approximation of a large class of implementation problems can be done utilizing only dominance solvable games, despite their rarity.

The experimental literature on level-k thinking and cognitive hierarchies (see, e.g., Costa-Gomes, Crawford, and Broseta, 2001; Camerer, Ho, and Chong, 2004) suggests limited ability of individuals to go through more than two iterations. Fudenberg and Liang (2019) conduct Amazon Mechanical Turk (MTurk) experiments using 200 two-player 3×3 games with payoffs determined uniformly at random.

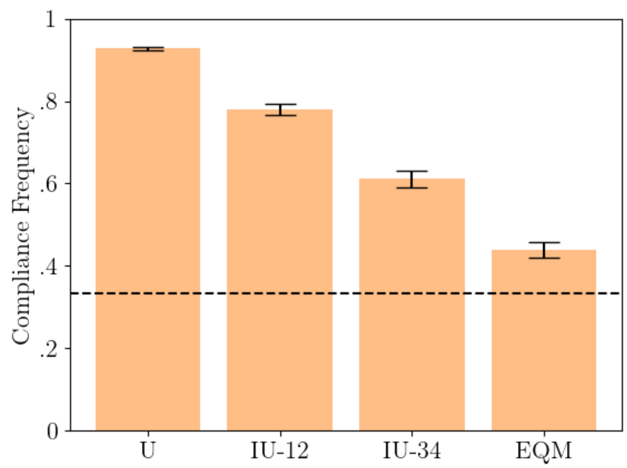


Figure 1: Frequency of Row's *undominated* decisions U in games with at least one Row's dominated action, *iteratively undominated* decisions IU-12 (IU-34, respectively) in *solvable* games with 1 or 2 (3 or 4, respectively) rounds that Row needs to identify her equilibrium decision, *equilibrium* decisions EQM in *non-solvable* games with one Nash equilibrium that is also pure

An analysis of their data reveals that solvable games requiring more iterations to reach equilibrium result in more frequent deviations from equilibrium decisions. Figure 1 demonstrates that compliance with equilibrium decisions for the row player (or simply Row) is high in *solvable* games in which Row needs one or two rounds to find her equilibrium decision, but significantly lower in *solvable* games in which Row needs to perform at least three iterations or *non-solvable* games with exactly one Nash equilibrium that is pure.[2] All these results suggest that dominance solvability alone may not suffice for "simplicity" of a game.

Conditional on a game being dominance solvable, we look at the number of iterations required to complete the elimination process. We show that this number is large, converging to the maximum possible as the number of actions of at least one of the players grows. As action sets expand, "simple" games become rare—they are unlikely to be dominance solvable and, even when they are, they likely require tremendous sophistication of players to reach an equilibrium outcome. Our results also open the door to questions regarding the features of dominance solvable games required to approximate various allocation objectives. Indeed, Katok, Sefton, and Yavas (2002) illustrate the limitations of virtual implementation in the lab due to the limited number of dominance iterations participants perform.

The number of actions players have to consider when playing a game may also serve as a complexity proxy. Even without dominance solvability, iterated elimination of strictly dominated actions may still be effective in simplifying a game if the set of actions surviving it is relatively small. We show that whether this is the case depends on the relative number of actions each player has in the underlying game. For $2 \times n$ games, the number of surviving actions for the second player has a mean of approximately $\log n$ and is asymptotically normally distributed. Furthermore, for $m \times n$ games with relatively small $m = o(\log n)$, the proportion of surviving actions for the second player converges to zero asymptotically. This provides a silver lining to our previous results—$m \times n$ games with relatively small m are dramatically simplified after our elimination process. The results are less rosy when the first player has more actions. We show that in $m \times n$ games with $m = \log_2 n + \omega(1)$, almost all actions survive the iterative deletion process, as n grows.[3]

We present all of our results for two-player games. We show, however, that results become even more stark for more than two players. In particular, the likelihood of dominance solvability declines and few actions are eliminated, even for three-player games.

Dominance solvability is so fundamental in game theory. Why don't we have these characterizations in our canon of knowledge already? We suspect there are two reasons. The first is that we often focus on games with very particular features in terms of complementarities, externalities, etc. At times, this focus is due to forces derived from particular applications. At times, this is done for tractability. At the very least, our results illustrate how special certain classes of games may be. The second reason is that our analysis requires some fairly

[2]We are grateful to Drew Fudenberg and Annie Liang for sharing their data with us.

[3]We write $f(n) = \omega(g(n))$ if $g(n) = o(f(n))$.

recent results in combinatorics. At its heart is the observation that the distribution of surviving actions after one iteration of elimination is closely linked to Stirling numbers of the first kind.[4] The asymptotics of these numbers' distributions, which we utilize, have been discovered only over the last couple of decades.

1.2. Literature Review. Dominance solvability was first introduced by Moulin (1979) as a weakening of strategy proofness in the context of voting.[5] Since then, dominance solvability has been studied in a variety of applications, ranging from auctions (see Azrieli and Levin, 2011, and references therein), to oligopolistic competition (Börgers and Janssen, 1995), to global games (Carlsson and van Damme, 1993).[6]

Due to the perceived simplicity of dominance solvable games, market designers have found them appealing for the implementation of various allocation problems. We already mentioned the important work on virtual implementation stemming from Abreu and Matsushima (1992a). Recently, concern for robust implementation has also directed attention to the potential usefulness of dominance solvable games, see e.g. Bergemann and Morris (2009). This literature rarely considers the number of iterations required to reach an equilibrium in dominance solvable games.[7] Nonetheless, as mentioned, the experimental literature suggests that implementation requiring multiple iterations may not generate the rationalizable outcomes, see Sefton and Yavas (1996) and Katok, Sefton, and Yavas (2002). In a lively discussion between Glazer and Rosenthal (1992) and Abreu and Matsushima (1992b), the latter suggest one solution: "agents can simply be educated about how the mechanism is solved!" Our results imply that it may be useful more often than not.

It is rare for experimental designs to allow participants to interact through randomly-determined games. Nonetheless, the literature on cognitive sophistication suggests that most individuals cannot perform many iterations, not without substantial experience (see Nagel, 1995; Costa-Gomes, Crawford, and Broseta, 2001; Camerer, Ho, and Chong, 2004).

[4]The k-th Stirling number of the first kind for a set of size n captures the number of permutations of n items with precisely k cycles. For a rich discussion of applications of these numbers, see Stanley (2011).

[5]As Moulin himself states, the guiding principle underlying dominance solvability was presented in earlier work by Farquharson (1969) and Brams (1975).

[6]Our study is also closely related to notions of rationalizability (Bernheim, 1984; Pearce, 1984) when one considers only pure strategies. Our analysis of surviving actions is then related to the rationalizable action set.

[7]An exception is Kartik, Tercieux, and Holden (2014), who consider agents with a small taste for honesty and characterize social choice functions that can be implemented by a simple mechanism in two rounds of iterated deletion of strictly dominated strategies.

As already mentioned, Fudenberg and Liang (2019) do conduct some experiments on two-player 3×3 games with payoffs determined uniformly at random. Our analysis of their data yield similar conclusions.

From a theoretical perspective, our paper is related in spirit to McLennan (2005) who, like us, considers random games. He analyzes the number of Nash equilibria, which rely on cardinal realizations of utilities, and uses very different methodology.

In terms of techniques, we use recent results in combinatorics by Hammett and Pittel (2008) and Hwang (1995, 1998). For recent reviews of results from that literature, see Stanley (2011, 2015). We hope the methodology we introduce can be useful for other, related problems.

2. The Model

2.1. Random Games. Consider a non-cooperative, simultaneous-move, one-shot game of complete information with two players, Row and Column. Row has m actions $[m] = \{1, 2, \dots, m\}$ and Column has n actions $[n] = \{1, 2, \dots, n\}$, where m, n are positive integers. Let $R = (r_{ij}) \in \mathbb{R}^{m\times n}$ and $C = (c_{ij}) \in \mathbb{R}^{m\times n}$ denote the $m \times n$ Row's and Column's payoff matrices respectively. We can represent this normal-form game by a bimatrix of the form

$$(R, C) = \begin{pmatrix} r_{11}, c_{11} & r_{12}, c_{12} & \dots & r_{1n}, c_{1n} \\ r_{21}, c_{21} & r_{22}, c_{22} & \dots & r_{2n}, c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1}, c_{m1} & r_{m2}, c_{m2} & \dots & r_{mn}, c_{mn} \end{pmatrix}.$$

In order to study the general properties of this class of games, we assume all payoffs are randomly generated. Since dominance solvability hinges on ordinal comparisons alone, we can focus on the randomness of payoff rankings, abstracting from the underlying cardinal payoff distributions.[8] In that sense, our analysis is "distribution-free." Let S_m denotes the symmetric group of permutations of $[m]$. We maintain the following *ordinal randomness assumption* throughout our analysis:

1. for each $j \in [n]$, Row's payoffs $r_{\cdot j}$ are uniform on S_m;[9]

[8]We ignore indifferences, which would arise with measure 0 for any continuous distribution of payoffs.

[9]Stanley (2015) discusses applications of random permutations in different scientific areas.

2. for each $i \in [m]$, Column's payoffs $c_{i\cdot}$ are uniform on S_n;

3. random permutations $\{r_{\cdot j}, c_{i\cdot}\}$, $i \in [m]$ and $j \in [n]$, are mutually independent.

In other words, for each fixed action of Row or Column, Column's or Row's ordinal rankings over its actions are uniform, and all ordinal rankings are mutually independent.[10]

Let $G(m, n)$ denote the corresponding *random game*.

2.2. Three Dimensions of Pure-Strategy Strict Dominance. In this paper, we examine the general properties of random games related to pure-strategy strict dominance. An action is *pure-strategy strictly dominated* if it always yields a worse outcome than some other action, regardless of other players' actions. If an action is not pure-strategy strictly dominated, it is called *pure-strategy strictly undominated*. An action is *strictly dominant* if all alternative actions are strictly dominated.

The elimination procedure that iteratively discards of all pure-strategy strictly dominated actions until there is no pure-strategy strictly dominated action is called *iterated elimination of pure-strategy strictly dominated actions*. We also call rounds of this elimination procedure *iterations*.[11] If by iterated elimination of pure-strategy strictly dominated actions there is only one action left for each player, the game is called a *pure-strategy strict-dominance solvable* game. To simplify the terminology in this paper, we will often omit the "pure-strategy" preamble.

Our analysis focuses on the following *three dimensions of strict dominance* for any random game $G(m, n)$. First, we ask how common strict-dominance solvable games are. We address this question by studying the *probability of strict dominance solvability*, denoted by $\pi(m, n)$. Second, we ask how "complex" strict-dominance solvable games are. We use the number of iterations required conditional on strict-dominance solvability as our complexity measure for strict-dominance solvable games. We call that measure the *conditional number of iterations* and denote it by $I(m, n)$. Last, we inspect the complexity of games surviving iterated elimination of pure-strategy strictly dominated actions. As a complexity measure for surviving games, we analyze the *number of surviving actions* after the iterated procedure, which we denote by $S^R(m, n)$ for Row and $S^C(m, n)$ for Column.

[10] Distributions matter if one allows for mixed strategies, which we do not. We return to this point in the concluding section of the paper.

[11] For finite games, the order in which pure-strategy strictly dominated actions are eliminated does not matter. To define the number of iterations, we suppose that at each iteration (or round) of the elimination procedure all players delete all pure-strategy strictly dominated actions.

As a by-product of our analysis, we also examine the number of strictly undominated actions denoted by $U^R(m,n)$ for Row and $U^C(m,n)$ for Column. It provides insights on the likelihood of games with a dominant-strategy equilibrium, where $U^R(m,n)$ and $U^C(m,n)$ are singletons.

3. Two Actions for One Player: Full Characterization

In this section, we fix the number of Row's actions to $m = 2$ and vary the number of Column's actions n.[12] Note that if $m = 1$, all realized games are dominance solvable within one iteration. Therefore, the first non-trivial case corresponds to $m = 2$.

For $m = 2$, we provide a full characterization. We first connect the distribution of the number of strictly undominated Column's actions to the *unsigned Stirling numbers of the first kind* that arise in various enumeration problems. We then use combinatorial techniques and probabilistic methods to obtain closed-form expressions for distributions of all variables of interest and examine the asymptotic behavior when $n \to \infty$.

Three insights emerge. First, dominance solvability is scarce—the probability $\pi(2,n)$ is asymptotically proportional to $n^{-1/2}$ and monotonically converges to zero as $n \to \infty$. Second, even seemingly simple $2 \times n$ dominant solvable games are complex. To be more specific, the conditional number of iterations exceeds 2 for 2×3 games and monotonically converges to the maximum number of 3. Third, however, surviving games are relatively simple. In fact, the proportion of surviving Column's actions converges to zero as $n \to \infty$ and Column can asymptotically proceed with one iteration alone.[13]

3.1. Undominated Actions. Because there are only two actions for Row, the distribution of her number of strictly undominated actions is straightforward. Note first that one action is strictly dominated by another action with probability $\left(\frac{1}{2}\right)^n$, where n is the number of Column's actions. In addition, there are two, mutually exclusive, ways to choose a strictly dominated action. Thus, Row has one strictly undominated action with probability $\left(\frac{1}{2}\right)^{n-1}$.

It is worth noting that we cannot proceed with the same argument for Column with n

[12] Due to symmetry, if we instead fix the number of Column's actions, the analysis is identical.

[13] In addition, the number of surviving Column's actions is asymptotically normal with expectation that goes to infinity.

actions and the payoff matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} \end{pmatrix},$$

where rows $\{c_{1\cdot}, c_{2\cdot}\}$ are *i.i.d.* uniform on S_n. One action is strictly dominated by another with probability $\frac{1}{4}$ in isolation. However, there are many ways by which one action can be dominated by various others, and they are no longer mutually exclusive.

We employ combinatorial techniques to avoid these complications. Because we care only about the number of undominated actions and not their labels, we have one degree of freedom when using two random permutations. Therefore, we can set either of $\{c_{1\cdot}, c_{2\cdot}\}$ to any fixed permutation. Without loss of generality, we fix $c_{1\cdot} = e_n \equiv (1, 2, \dots, n)$ and focus on

$$C = \begin{pmatrix} 1 & 2 & \dots & j & \dots & n \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2n} \end{pmatrix},$$

where $c_{2\cdot}$ is uniform on S_n. Formally, our notation above with two rows that are *i.i.d.* uniform on S_n is equivalent to the two-row notation with one fixed row and another drawn uniform from S_n.

Several insights follow immediately in this simplified problem. First, the n-th action is always strictly undominated. Furthermore, for any $1 \le j \le n-1$, the j-th action is strictly undominated if and only if $c_{2j} > c_{2i}$, which occurs with probability $\frac{1}{n-j+1}$. Thus, because of linearity of expectations, the expected number of Column's undominated actions is H_n, where $H_n \equiv 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the n-th harmonic number. Since $H_n \sim \log n$, asymptotically, the proportion of Column's undominated actions is negligible. Nonetheless, we have many dependencies that do not allow us to compute the desired distribution directly.

Our analysis relies on the recursive structure of the simplified problem. There are $n!$ combinations in total for $c_{2\cdot}$. Let $f(n, k)$ denote the number of combinations corresponding to exactly k of Column's actions being strictly undominated, $k \in [n]$. There are two relevant cases. If $c_{21} \in [n-1]$, then Column's first action is strictly dominated and we need to have k undominated actions among the remaining $(n-1)$ actions. If $c_{21} = n$, Column's first action is strictly undominated and we need to have $k-1$ undominated actions among the remaining $(n-1)$ actions. Therefore,

$$f(n, k) = (n-1)f(n-1, k) + f(n-1, k-1).$$

This expression corresponds to the recurrence relation of the unsigned (or signless) Stirling numbers of the first kind commonly denoted by $c(n,k)$:

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1), \quad n,k \geq 1,$$

with the initial conditions $c(n,k) = 0$ if $n < k$ or $k = 0$, except for $c(0,0) = 1$. Lemma 1 formalizes the intuition stated above.

Lemma 1. *Consider a random game $G(2,n)$. Then, for any $n \geq 1$,*

1. $\Pr\left(U^R(2,n) = 1\right) = \dfrac{1}{2^{n-1}}$;
2. *for any $k \in [n]$,* $\Pr\left(U^C(2,n) = k\right) = \dfrac{c(n,k)}{n!}$.

The combinatorics literature offers various definitions and interpretations for the unsigned Stirling numbers of the first kind. The original definition of the unsigned Stirling numbers of the first kind $c(n,k)$ is algebraic. Namely, they are the coefficients $c(n,k)$ in the expansion of the rising factorial

$$x^{(n)} \equiv x(x+1)\dots(x+n-1) = \sum_{k=0}^{n} c(n,k)x^k.$$

We use this definition to find the probability of dominance solvability in Proposition 1 below. There are various other interpretations. For instance, $c(n,k)$ corresponds to the number of permutations $\sigma \in S_n$ with exactly k cycles.[14]

3.2. Dominance Solvability. In order to express the probability of dominance solvability, we need to introduce additional notation. The double factorial of a positive integer n, denoted by $n!!$, is defined as the product of all the integers from 1 up to n that have the same parity (odd or even) as n.[15] In addition, let $W(n)$ denote the so-called Wallis ratio (Qi and Mortici,

[14] There are other enumerative problems involving the unsigned Stirling numbers of the first kind (Stanley, 2011). For instance, $c(n,k)$ captures the number of unordered increasing trees on $n+1$ vertices for which the root has k successors. In addition, $c(n,k)$ enumerates permutations of $\sigma \in S_n$ with exactly k records (or left-to-right maxima). Interestingly, despite having the same enumerative properties, the permutation class corresponding to k cycles, that corresponding to k records, and that associated with k strictly undominated actions are pairwise different.

[15] By definition, $(2n-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdot \ldots \cdot (2n) = n! \cdot 2^n$.

2015) defined as

$$W(n) \equiv \frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)},$$

where $\Gamma(x)$ is the gamma function with $\Gamma(1/2) = \sqrt{\pi}$.

Proposition 1 provides analytical formulas for the probability of dominance solvability.

Proposition 1. *Consider a random game $G(2,n)$. Then,*

1. *for any $n \geq 1$, $\pi(2,n) = 2W(n) = \dfrac{(2n-1)!!}{2^{n-1} \cdot n!}$;*
2. *$\pi(2,n)$ is strictly decreasing in n;*
3. *$\lim_{n\to\infty} n^{1/2} \cdot \pi(2,n) = \dfrac{2}{\sqrt{\pi}}$.*

The formal proof appears in the Appendix. Intuitively, we derive the exact formula for $\pi(2,n)$ as follows. Recall that the order in which strictly dominated actions are eliminated does not matter. There are n possible mutually exclusive events corresponding to the number k of strictly undominated actions for Column, $k \in [n]$, each occurring with probability $\frac{c(n,k)}{n!}$ respectively. The induced $2 \times k$ game, derived from eliminating all of Column's dominated actions, is strict-dominance solvable if and only if Row has exactly one strictly undominated action. This occurs with probability $\left(\frac{1}{2}\right)^{k-1}$ since Row's and Column's payoffs are independent. By summing over all possible cases $k \in [n]$, remembering that either of Row's two actions can survive, and using the algebraic definition of $c(n,k)$ together with various well-known identities, we get the desired expression. The monotonicity of $\pi(2,n)$ follows from the identity $\Gamma(x+1) = x\Gamma(x)$. The asymptotic characterization follows from Stirling's formula applied to the gamma function.

3.3. Conditional Iterations. There is exactly one iteration conditional on $G(2,n)$ being strict-dominance solvable if and only if both Row and Column have strictly dominant actions. Thus, using the identity $\Gamma(n+1) = n\Gamma(n)$, we have:

$$\Pr(I(2,n)=1) = \left(\frac{1}{2^{n-1}} \cdot \frac{1}{n}\right) \cdot \frac{1}{\pi(2,n)} = \frac{\sqrt{\pi}}{2^n} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)} \sim \sqrt{\pi} \cdot \frac{1}{2^n \cdot n^{1/2}}.$$

Now, there are exactly two iterations conditional on $G(2,n)$ being strict-dominance solvable if and only if either Row or Column have a dominant action, not both. That is,

$$\Pr(I(2,n)=2) = \left(\frac{1}{2^{n-1}} + \frac{1}{n} - \frac{1}{2^{n-2}} \cdot \frac{1}{n}\right) \cdot \frac{1}{\pi(2,n)} = \frac{n+2^{n-1}-2}{2^n} \cdot \sqrt{\pi} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)} \sim \frac{\sqrt{\pi}}{2} \cdot \frac{1}{n^{1/2}}.$$

Finally, there are three conditional iterations in the remaining case corresponding to the alternating elimination procedure starting from Column:

$$\Pr(I(2,n)=3) = 1 - \frac{n+2^{n-1}-1}{2^n} \cdot \sqrt{\pi} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)} \sim 1 - \frac{\sqrt{\pi}}{2} \cdot \frac{1}{n^{1/2}}.$$

We show in Lemma 4 in the Appendix that $\Pr(I(2,n)=1)$ is monotonically, and exponentially, decreasing to zero as the number of Column's actions n goes to infinity. In particular, it is unlikely for games to be solvable in strictly dominant actions. In fact, as the derivation above suggests, it is rare to have a strictly dominant action even for only one of the players. Formally, $\Pr(I(2,n)=2)$ is also monotonically decreasing, albeit not exponentially, to zero. Therefore, asymptotically, the more pervasive manner by which dominance solvability is achieved involves the maximum of three elimination iterations, where Column is the first to eliminate actions. Proposition 2 summarizes this discussion by focusing on the expected number of conditional iterations.

Proposition 2. *Consider a random game $G(2,n)$. Then,*

1. $\mathbb{E}\left[I(2,n)\right] = 3 - \frac{n+2^{n-1}}{2^n} \cdot \sqrt{\pi} \cdot \frac{\Gamma(n)}{\Gamma(n+1/2)}$*;*
2. $\mathbb{E}\left[I(2,n)\right]$ *is strictly increasing in n;*
3. $\lim_{n\to\infty} n^{1/2} \cdot (3 - \mathbb{E}\left[I(2,n)\right]) = \frac{\sqrt{\pi}}{2}.$

As mentioned in the Introduction, experimental evidence suggests individuals' limited ability to go beyond two iterations. Proposition 2 then implies that most $2 \times n$ games that are dominance solvable may be de-facto challenging to reason through. As we will soon show, this point becomes even starker when both players have a substantial number of actions.

3.4. Surviving Actions. Row has exactly one action surviving iterated elimination of strictly dominated actions if and only if the considered game is strict-dominance solvable. Therefore, we immediately have

$$\Pr\left(S^R(2,n)=1\right) = \pi(2,n), \quad \Pr\left(S^R(2,n)=2\right) = 1-\pi(2,n), \quad \text{and} \quad \mathbb{E}\left[S^R(2,n)\right] = 2-\pi(2,n),$$

and both comparative statics and asymptotic features follow directly from Proposition 1.

As for Column, by similar arguments we obtain

$$\Pr\left(S^C(2,n)=1\right)=\pi(2,n).$$

For any $k \neq 1$, $k \in [n]$, Column has exactly k surviving actions if and only if it has exactly k undominated actions and the considered game is not strict-dominance solvable, so that

$$\begin{aligned}\Pr\left(S^C(2,n)=k\right) &= \Pr\left(U^C(2,n)=k\right)\cdot\Pr\left(S^R(2,n)\neq 1 \mid U^C(2,n)=k\right)\\ &= \Pr\left(U^C(2,n)=k\right)\cdot\Pr\left(U^R(2,k)\neq 1\right)=\frac{c(n,k)}{n!}\cdot\left(1-\frac{1}{2^{k-1}}\right),\end{aligned}$$

where the second equality follows from independence between Row's and Column's payoffs.

As for Column, it follows from Proposition 1 that, asymptotically, Row has nothing to eliminate, so that Column can eliminate actions only in his first iteration. Intuitively, then, the difference between the number of Column's strictly undominated actions and the number of Column's surviving actions vanishes asymptotically.

More formally, the distribution of the number of Column's surviving actions is similar to that pertaining to strictly undominated actions with two exceptions. First, for any $k \neq 1$, $k \in [n]$, the corresponding probabilities are discounted by $\left(1-\frac{1}{2^{k-1}}\right)$ with smaller discounts for larger k. Second, there is a spike at $k=1$ in the distribution of the number of Column's surviving actions that corresponds to the probability of strict-dominance solvability. Indeed, note that for any $n \geq 2$, because $\Pr\left(U^R(2,2)=1\right)=1/2$, we have $\left(S^C(2,n)=1\right) > \Pr\left(S^C(2,n)=2\right)$.

For any $n \geq 1$, the sequence of numbers $c(n,k)$, $k=0,1,\ldots,n$, is log-concave[16] and, hence, unimodal (Stanley, 2011). In addition, the signless Stirling number $c(n,k)$ is maximized at $k(n)$ that is either $\lfloor H_n \rfloor$ or $\lceil H_n \rceil$. That is, $k(n) \sim \log n$ asymptotically. Results by Hwang (1995) suggest that, although for any fixed $k \neq 1$, $k \in [n]$ the corresponding probability $\Pr\left(S^C(2,n)=k\right)$ converges to zero faster than $\Pr\left(S^C(2,n)=1\right)=\pi(2,n)$—the spike mentioned before—the *mode* of this distribution corresponding asymptotically to $k(n) \sim \log n$ converges to zero slower. Lemma 5 in the Appendix formalizes this claim. In fact, results from probabilistic combinatorics also suggest that the distribution induced by $\frac{c(n,k)}{n!}$ is asymptotically normal (Gontcharoff, 1944; Hwang, 1998). Our discussion then suggests that the number of surviving Column's actions is asymptotically normal. Proposition 3

[16] A sequence $a=(a_0,a_1,\cdots,a_n)$ of nonnegative real numbers is log-concave if $a_k^2 \geq a_{k-1}a_{k+1}$ for any $k \in [n-1]$.

formalizes this intuition.

Proposition 3. *Consider a random game $G(2,n)$. Then,*

$$\Pr\left(S^C(2,n) - \mathbb{E}\left[S^C(2,n)\right] \leq x \cdot \sqrt{\mathrm{Var}\left[S^C(2,n)\right]}\right) = \Phi(x) + \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right),$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution,

$$\mathbb{E}\left[S^C(2,n)\right] = \log n + \gamma + o(1), \quad \textit{and} \quad \sqrt{\mathrm{Var}\left[S^C(2,n)\right]} = \sqrt{\log n} - \frac{\pi^2 - 6\gamma}{12\sqrt{\log n}} + o\left(\frac{1}{\sqrt{\log n}}\right).$$

This result can be interpreted as a variant of the central limit theorem for dependent variables. Indeed, the number of Column's surviving actions is the sum of indicator random variables, each corresponding to whether a given action survives or not. While there are many non-trivial dependencies among these variables, we still have an asymptotic normality for the aggregate number.

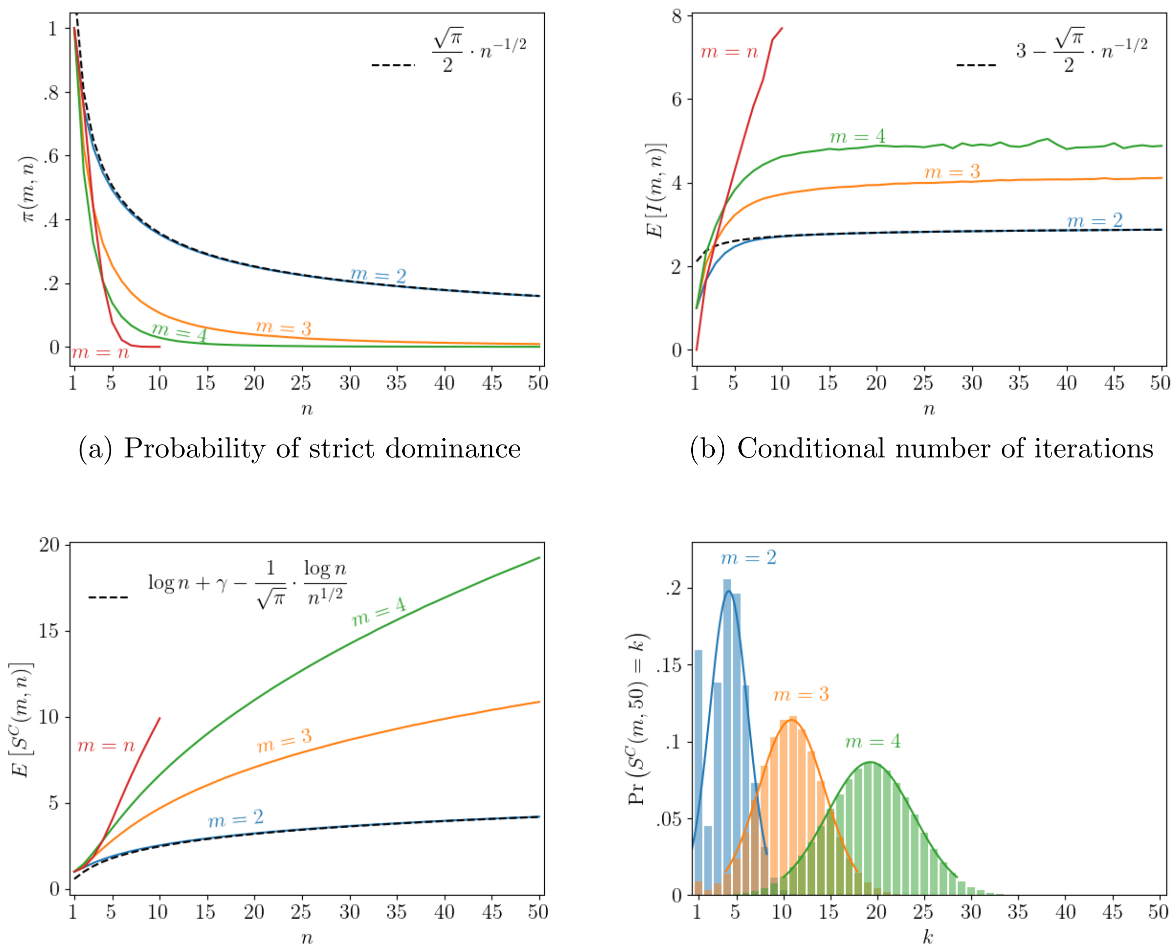
The formal proof of Proposition 3 appears in the Appendix. In rough terms, it follows similar lines to those appearing in the analysis of Hwang (1998).[17] It uses the Berry-Esseen inequality (Petrov, 1975) stated in terms of characteristic functions to find convergence rates. Namely, using the algebraic definition of $c(n,k)$, we compute the characteristic function of the number of surviving Column's actions and compare it to the characteristic function of the standard normal distribution.

Figure 2 summarizes our discussion in this subsection, when focusing on the $m = 2$ curves. The panels of the figure depict the different objects we analyze for random games varying in size, $n = 1, ..., 50$: the probability of dominance solvability in panel (a), the expected number of conditional iterations in panel (b), and the expectation and distribution of the number of surviving Column actions in panel (c). In addition to their exact values, we also depict the asymptotic behavior analytically described in our results. As can be seen, our asymptotic characterizations provide remarkably close approximations for $2 \times n$ games in which $n > 5$.

4. Arbitrary Action Sets: General Analysis

By the symmetry with respect to players' names, we fix the number of Column's actions to n and vary the number of Row's actions $m \leq n$, as a function of n. This is without loss

[17] Since we have a spike and discounted probabilities, the problem does not belong to the exp-log class that Hwang (1998) studies. Therefore, we cannot use his results directly.



(c) Surviving actions

Figure 2: Three dimensions of dominance solvability

of generality.

In contrast to $m = 2$, the general problem introduces novel enumerative issues that have not yet been studied in the combinatorics literature. We employ mostly probabilistic methods to obtain closed-form expressions for some variables of interest and study their asymptotic patterns. When it is hard to obtain closed-form expressions, we use computer simulations instead.[18]

Our general analysis suggests several main insights. First, dominance solvability is rare—for $m \leq n$, $\pi(m,n) = n^{-\Theta(m)}$ and it converges to zero as $n \to \infty$ at convergence rates that

[18]We generate games by using *i.i.d.* $\mathcal{U}(0,1)$ random draws. This is without loss of generality because our analysis is distribution-free. Due to computational limitations, we typically use $S = 10^6$ simulations.

increase in m. Second, for relatively small m as a function of n, games are greatly simplified after the first iteration—for $m = o(\log n)$, the proportion of Column's undominated actions converges to zero. Otherwise, iterated elimination becomes less potent in altering the game players need to consider. In particular, if $m = \log_2 n + \omega(1)$, almost all actions remain for both players. When $m = 2\log_2 n + \omega(1)$, players cannot eliminate any action at all. Second, even though particular games may be greatly simplified after the first iteration, subsequent iterations are no longer effective— one iteration is asymptotically sufficient. Finally, the conditional number of iterations is large even for relatively small games, e.g., it exceeds 3 in expectation even for 4×4 random games.

4.1. Enumerative Issues. For general $m \times n$ games, there are two main complications that need to be handled.

First, there are even more sophisticated interdependent ways for a given action to be eliminated. These dependencies break down the recursive structure that we previously used to find the distribution of the number of Column's undominated actions for $m = 2$. As an illustration, we discuss the case of $m = 3$ in the Appendix.

The second enumerative issue is that the induced remaining games after each iteration of elimination have non-trivial conditional payoff distributions—if, for example, k actions are undominated for Column after the first iteration, those actions are statistically linked. In other words, the ordinal randomness assumption no longer holds. This was not a problem in the 2 by n case, where we dealt with a maximum of three iterations. However, it becomes a serious problem in the general case since the maximum number of possible iterations grows linearly.[19] Consequently, we mostly rely on simulations to compute the conditional number of iterations, which appears to be the most challenging object of our analysis.

4.2. Undominated Actions. In what follows, we provide insights derived by avoiding these two complications. Without loss of generality, we focus on $m \leq n$.

Intuitively, it is harder for Column to eliminate his actions as the number of Row's actions becomes larger. We therefore have the following lemma.

Lemma 2. *Consider a random game $G(m,n)$. Then, for any $m \geq 1$, $n \geq 2$, and $k \in [n]$,*

$$\Pr\left(U^C(m+1,n) \leq k\right) \leq \Pr\left(U^C(m,n) \leq k\right) \text{ with a strict inequality for some } k.$$

[19]Note that the maximum number of iterations for the m by n game is $2m-1$ when $m < n$ and $2m-2$ if $m = n$.

In other words, $U^C(m+1,n)$ first-order stochastically dominates $U^C(m,n)$.

In fact, we can use the linearity of expectation to compute the expected number of Column's undominated actions. By symmetry, the probability that a given Column's action is undominated in $G(m,n)$ is independent of its label. Therefore, the expected proportion of Column's undominated actions $\frac{\mathbb{E}\left[U^C(m,n)\right]}{n}$ is also the probability that the first of Column's actions is undominated. In our proof, we obtain the recurrence relation for this probability, and hence for $\mathbb{E}\left[U^C(m,n)\right]$, by considering n mutually exclusive events, each corresponding to $c_{m1} = k$ for some $k \in [n]$. Any such event occurs with probability $\frac{1}{n}$. If $c_{m1} = k$, the first action is undominated if and only if it is undominated in the reduced $(m-1) \times (n-k+1)$ game formed by removing all columns j with $c_{mj} < k$ and the last row. This event occurs with probability $\frac{\mathbb{E}\left[U^C(m-1,n-k+1)\right]}{n-k+1}$. By summing over all possible cases $k \in [n]$, we obtain the desired recurrence relation, and its consequences, stated in the proposition below.

Proposition 4. *Consider a random game $G(m,n)$. Then, for any $m,n \geq 2$,*

1. $\mathbb{E}\left[U^C(m,n)\right] = \sum_{k=1}^{n} \frac{\mathbb{E}\left[U^C(m-1,k)\right]}{k}$;
2. *it is component-wise strictly increasing;*
3. *it satisfies* $\frac{(\log n)^{m-1}}{(m-1)!} \leq \mathbb{E}\left[U^C(m,n)\right] \leq \sum_{k=0}^{m-1} \frac{(\log n)^k}{k!}$.

Next, we analyze asymptotic estimates of the corresponding two bounds in part 3 of the proposition. We immediately get $\sum_{k=0}^{m-1} \frac{(\log n)^k}{k!} = \frac{(\log n)^{m-1}}{(m-1)!}(1+o(1))$ if and only if $\frac{(\log n)^{m-2}}{(m-2)!} = o\left(\frac{(\log n)^{m-1}}{(m-1)!}\right)$. This in turn holds if and only if $m = o(\log n)$, or when m is sufficiently small relative to n.

The upper bound $\sum_{k=0}^{m-1} \frac{(\log n)^k}{k!}$ is the beginning of the Taylor expansion of $e^{\log n} = n$, so that it is at most n. For every fixed n, it converges to n as m goes to infinity. In fact, we can rewrite

$$\sum_{k=0}^{m-1} \frac{(\log n)^k}{k!} = n \cdot \sum_{k=0}^{m-1} \left.\frac{\lambda^k e^{-\lambda}}{k!}\right|_{\lambda=\log n} = n \cdot \Pr\left(\text{Poisson}(\log n) \leq m-1\right),$$

where Poisson(λ) is a Poisson random variable with parameter $\lambda > 0$ corresponding to both mean and variance. Therefore, for $m \geq \log n + C \cdot \sqrt{\log n}$ with some large constant $C > 0$, the upper bound is very close to n and it converges to n as C increases indefinitely. Choi

(1994) proved that the median of Poisson(λ) satisfies $\lambda - \log 2 \leq \text{med}(\lambda) < \lambda + \frac{1}{3}$ and these inequalities are sharp. For that reason, the upper bound is larger than $\frac{n}{2}$ for any $m \geq \log n + \frac{4}{3}$.

Similarly, the lower bound in part 3 of the proposition corresponds to

$$\frac{(\log n)^{m-1}}{(m-1)!} = n \cdot \left.\frac{\lambda^{m-1} e^{-\lambda}}{(m-1)!}\right|_{\lambda = \log n} = n \cdot \Pr\big(\text{Poisson}(\log n) = m-1\big).$$

The modes of Poisson(λ) are $\text{mode}(\lambda) = \{\lfloor \lambda \rfloor, \lceil \lambda \rceil - 1\}$. By Stirling's formula, $\frac{(\log n)^{m-1}}{(m-1)!n}$ is roughly $\frac{1}{\sqrt{2\pi m}}$ asymptotically for $n = e^{m-1}$, that is $m = \log n + 1 \approx \text{mode}(\log n) + 1$.

As m gets large relative to n, this lower bound converges to zero and hence becomes less useful. Intuitively, the probability that Poisson($\log n$) is equal to the given specific value $(m-1)$ becomes negligible. However, by considering unions, we obtain $\frac{\mathbb{E}\left[U^C(m,n)\right]}{n} \geq 1 - \frac{n-1}{2^m}$, since for any given Column's action, there are $n-1$ other actions that may strictly dominate it, each with probability $\frac{1}{2^m}$. Importantly, this bound is valid for large m.

In conclusion, we have:

$$\max\left\{\frac{(\log n)^{m-1}}{(m-1)!n}, 1 - \frac{n-1}{2^m}\right\} \leq \frac{\mathbb{E}\left[U^C(m,n)\right]}{n} \leq \frac{1}{n} \cdot \sum_{k=0}^{m-1} \frac{(\log n)^k}{k!}.$$

We summarize our discussion here with the following proposition. For relatively small $m = o(\log n)$, games are greatly simplified after the first iteration in the sense that the proportion of Column's undominated actions approaches zero. In contrast, for larger $m = \log_2 n + \omega(1)$, almost all actions remain. Furthermore, if $m = 2\log_2 n + \omega(1)$, Column is further limited in his ability to eliminate actions in the first iteration.

Proposition 5. *Consider a random game $G(m,n)$. Then, for any $m \leq n$, if*

1. $m = o(\log n)$, *then* $\frac{\mathbb{E}\left[U^C(m,n)\right]}{n} = o(1)$;
2. $m = \log_2 n + \omega(1)$, *then* $\frac{\mathbb{E}\left[U^C(m,n)\right]}{n} = 1 - o(1)$;
3. $m = 2\log_2 n + \omega(1)$, *then* $\mathbb{E}\left[U^C(m,n)\right] = n - o(1)$.

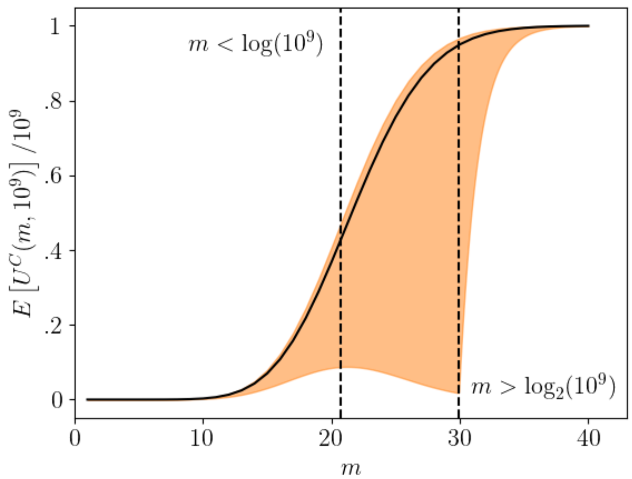


Figure 3: Expected proportion of Column's undominated actions $\mathbb{E}\left[U^C(m,10^9)\right]/10^9$

For all intermediate m, Figure 3 indicates that the actual proportion of Column's undominated actions is very close to its upper bound $\frac{1}{n}\cdot\sum_{k=0}^{m-1}\frac{(\log n)^k}{k!}$. Based on our previous discussion about its asymptotic estimates, $\sum_{k=0}^{m-1}\frac{(\log n)^k}{k!}$ exceeds $\frac{n}{2}$ for $m \geq \log n + \frac{4}{3}$ and is already very close to n as $m \geq \log n + C\cdot\log n$ for some large $C > 0$. Therefore, we expect the actual proportion to be relatively large and close to one for the majority of intermediate cases.

Turning to Row, since we set $m \leq n$, from our previous discussion it immediately follows that $\mathbb{E}\left[U^R(m,n)\right] \geq m - \frac{m(m-1)}{2^n}$, which converges to m rapidly.

As an immediate corollary, the iterative elimination procedure does not succeed in altering action sets in games with $m = \log_2 n + \omega(1)$, namely when Row does not have a small action set. Furthermore, if $m = 2\log_2 n + \omega(1)$, both players cannot delete any action at all.

4.3. Surviving Actions. With respect to actions surviving iterated elimination, note first that for any $m, n \geq 2$, $S^C(m,n)$ trivially first-order stochastically dominates $U^C(m,n)$. Intuitively, there are many more restrictions for an action to survive the full sequence of iterations rather than be undominated in the first iteration.

In the previous section, we saw that the first iteration can significantly simplify games with $m = o(\log n)$, when Row has a small action set relative to the one of Column. It turns out that players cannot proceed beyond this first iteration asymptotically.

Proposition 6. *Consider a random game $G(m,n)$. Then, for any $m \leq n$,*

1. *if* $m = \mathcal{O}(1)$*, then* $\Pr\left(S^R(m,n) < m\right) = \mathcal{O}\left(n^{-\frac{m-1}{2}}\right)$*;*

2. $\Pr\left(S^R(m,n) < m\right) \leq m(m-1) \cdot \left(\frac{m}{n}\right)^{\frac{m-1}{4}}$.

Recall that in the 2 by n case, the probability of Row eliminating any of her actions coincides with the probability of strict-dominance solvability of order $\mathcal{O}\left(n^{-\frac{1}{2}}\right)$ for large n. In the m by n case with fixed $m \geq 3$ and large n, following Lemma 2, Column is more likely to have even more undominated actions. Therefore, Row has fewer opportunities to eliminate actions. This reasoning is formalized in Proposition 6 that states that Row has nothing to eliminate at all as the number of Column's actions increases indefinitely.

Proposition 6 implies that the difference between the number of Column's strictly undominated actions and the number of his surviving actions is negligible asymptotically. That is, one iteration is asymptotically sufficient.

4.4. Dominance Solvability. When considering the likelihood of dominance solvability, it is important to note that not all subgames of a strict-dominance solvable $m \times n$ game are strict-dominance solvable. Furthermore, in general, we cannot partition a non-random strict-dominance solvable game into two non-intersecting strict-dominance solvable subgames of given dimensions. That is, $\pi(m,n)$ is not component-wise sub-multiplicative. Nonetheless, we can pick *particular* subgames of any size that are dominance solvable:

Lemma 3. *Consider a strict-dominance solvable* $m \times n$ *game, where* m, n *are positive integers. Then, for any* $m' \in [m]$*,* $n' \in [n]$*, there exists a strict-dominance solvable* $m' \times n'$ *subgame.*

The main observation underlying this monotonicity result is that for a strict-dominance solvable $m \times n$ game, any subgame generated by eliminating one of the players' dominated actions would be dominant solvable as well.

As an immediate corollary, for any fixed $m \geq 2$, $\pi(m,n) \leq m \cdot \pi(m-1,n)$, and hence convergence rates are weakly increasing.

We can also apply Proposition 6 to examine the probability of strict-dominance solvability $\pi(m,n)$. Indeed, if a random game $G(m,n)$, $m \geq 2$, is strict-dominance solvable, then only one of Row's actions must survive. It follows that $\pi(m,n) = \Pr\left(S^R(m,n) = 1\right) \leq \Pr\left(S^R(m,n) < m\right)$, characterized in the previous proposition.

When Row has many actions, the iterative elimination procedure needs to eliminate plenty of actions, not just one. Proposition 7 obtains a bound for $n \times n$ games that is more accurate than the crude union bound, $2n(n-1) \times 2^{-n}$.

Proposition 7. *Consider a random game $G(n,n)$. Then, $\pi(n,n) \leq n^{-\left(\frac{1}{3}-o(1)\right)n}$.*

In our proof, we start the iterative deletion process and terminate it exactly when at least $\frac{n}{3}$ of Row's actions or at least $\frac{n}{3}$ of Column's are already eliminated. When restricted to Column's (Row's) actions not eliminated yet, any Row's (Column's) eliminated action must be dominated by her (his) action that is not yet deleted. This observation allows us to estimate the probability of elimination of at least $\frac{n}{3}$ rows or columns in $n \times n$ games by avoiding conditioning. It represents a more restrictive bound for the probability $\pi(n,n)$ of strict-dominance solvability.

Our results thus far imply the following general bounds on $\pi(m,n)$.

Proposition 8. *There exist $C_1, C_2 > 0$ such that for any $m \leq n$,*

$$n^{-(m-1)} \leq \pi(m,n) \leq C_1 \cdot n^{-C_2 m}.$$

The lower bound is obtained by noting that if Column has a strictly dominant action in a game, then this game is generically strict-dominance solvable.

To derive the upper bound, we consider two cases in turn. If $m \leq n^{0.9}$, the general inequality proved in Proposition 6 gives the desired bound. Otherwise, if $m \geq n^{0.9}$, we use our monotonicity lemma to get $\pi(m,n) \leq \binom{n}{m} \cdot \pi(m,m)$ and apply Proposition 7 to $\pi(m,m)$. With some algebraic manipulation, we get constants $C_1, C_2 > 0$ that work for both these cases.

4.5. Conditional Iterations. Due to the difficult statistical conditioning discussed before, we employ simulations to analyze the conditional number of iterations for $m \times n$ games with $m \geq 3$.

Figure 2 uses simulations to depict the objects of our analysis for $m \times n$, $m = 3, 4$ and $n \in [50]$, as well as $n \times n$ games, $n \in [10]$, with 10^6 simulations for each game size, in addition to exact values corresponding to the $m = 2$ case discussed in the previous section.[20]

For $m \times n$ games with $m > 2$, all qualitative conclusions resemble those derived for the $m = 2$ case. In particular, the probability of strict-dominance solvability converges to zero,

[20]For $n \times n$ games, we restrict n to be no larger than 10 for computational reasons.

albeit more rapidly, and both the number of conditional iterations as well as the number of surviving Column actions grow with m. The distribution of surviving actions appears approximately normal even when fixing n at 50.

Figure 2 also highlights the contrast between games in which both players' action sets expand and those in which one of the players has a fixed action set. The likelihood of dominance solvability $\pi(n, n)$ vanishes quickly, standing at less than 5% when $n \geq 7$, the number of conditional iterations $I(n, n)$ is increasing indefinitely and exceeds 3 starting from $n = 4$. The expected number of (pure-strategy) rationalizable actions coincides with the full action set even for small n, namely any $n \geq 5$.

5. Conclusions and Discussion

This paper provides a characterization of several features resulting from iterative elimination of strictly dominated actions in general random games. We show that "simple" games, ones that are dominance solvable in fairly few steps, are rather rare. Iterated elimination can help players simplify the game when at least one of the players has a small number of actions. However, as soon as both players expand their action sets, or more than two players interact, elimination of actions becomes less likely and iterative elimination loses its effectiveness in simplifying the game. From a technical perspective, we show the usefulness of several methods from the statical combinatorics. In particular, we show the role Stirling numbers of the first kind play in dominance solvability properties.

Many games studied in the Economics literature, both theoretically and empirically, are altered by iterated elimination. As an illustration, Fudenberg and Liang (2019) consider a sample of 3×3 games from the experimental literature and provide a comparison with a large collection of randomly-generated games. They suggest that experimental games tend to have more pure Nash equilibria and more rationalizable actions than random games. Games receiving attention in the literature are inspired by applications. It might be that real-world interactions tend to induce structure conducive to the effectiveness of iterated elimination. It would be interesting to expand our analysis to particular classes of games, say, ones with certain complementarities or externalities.

In what follows, we discuss several extensions and caveats to our study.

5.1. Many Players. Consider a random n-person game $G(m_1, m_2, \ldots, m_n)$, where player k has m_k actions, $k \in [n]$. According to our analysis, player k has the expected number of her undominated actions that equals $\mathbb{E}\left[U^k\left(m_1, m_2, \ldots, m_n\right)\right] = \mathbb{E}\left[U^C\left(\prod_{i\neq k} m_i, m_k\right)\right]$.

As the number of players grows, it is exponentially harder for players to eliminate any action at all. As an illustration, in a symmetric n-person game, where each player has m actions, one player's action dominates another with probability $2^{-m^{n-1}}$. By the union bound (Boole's inequality), the probability that players can eliminate any action is bounded above by $nm(m-1)\cdot 2^{-m^{n-1}}$ that converges to zero rapidly. While we consider here the case in which all players have the same number of actions, the same reasoning holds for asymmetric random games with several players as well.

Therefore, results pertaining to dominance solvability features are straightforward even for small games with at most three players and closely trace results for large two-player games in which players' action sets expand. First, getting strict-dominance solvability is challenging even for small games since it is unlikely players can eliminate any action at all. Second, surviving games are likely to coincide with the games prior to the process of elimination—iterated elimination is statistically ineffective at simplifying games. Last, since it is even more demanding for any player to eliminate many actions at each elimination round, we expect the conditional number of iterations to be large even in small games. Confirming our argument, Figure 4 shows that for random three-player games $G(2, n, m)$ with $G(2, n, 1)$ being equivalent to a two-player game $G(2, n)$, the conditional number of iterations is component-wise increasing and exceeds 3 even for small game dimensions.

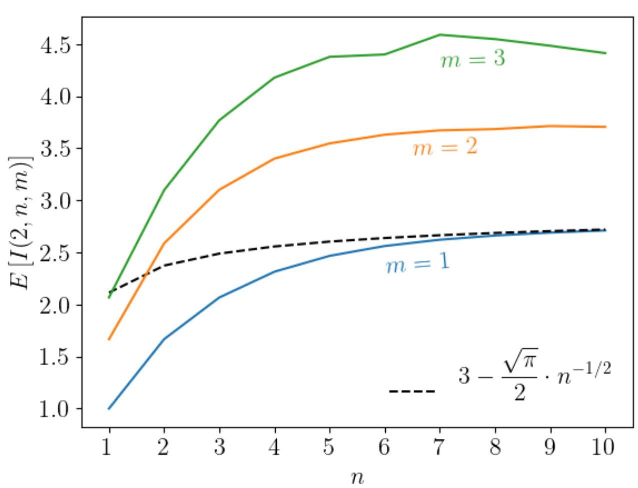


Figure 4: Conditional number of iterations $I(2, n, m)$ for $m = 1, 2, 3$ and $n = 1, 2, \ldots, 10$

Because we have qualitative results even for small games, asymptotics is less interesting for random games with many, at least three, players. However, similar arguments to those used in the paper can be exploited to show that convergence rates of the probability of strict-dominance solvability are much faster than for the two-player case.

Regarding comparative statics, because any random two-player $G(m_1, m_2)$ game can be represented as a n-player $G(m_1, m_2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}})$, we expect all qualitative results to generalize.

5.2. Mixed Strategies. Throughout this paper, we consider elimination of strictly dominated actions considering only *pure* actions. Allowing for mixed strategies introduces several new complications. First, the analysis is no longer distribution-free—cardinal, not only ordinal, payoff details matter. Consequently, purely combinatorial techniques cannot be used. Simulations are also more challenging computationally as all convex combinations of actions need to be taken into account when assessing dominance.

Admittedly, because mixed-strategies provide more flexibility, random games are more likely to be mixed-strategy strict-dominance solvable. We view the analysis of mixed-strategy strict-dominance solvability as an interesting direction for future research.

It is worth noting, however, that despite their mathematical flexibility, experimental evidence suggests that mixed strategies are more cognitively demanding (see Erev and Roth, 1998; Shachat, 2002).

Appendix – Proofs

Proof of Lemma 1. Let's prove two statements in turn.

1. By taking into account that either of Column's two actions can remain,

$$\Pr\left(U^R(2,n)=1\right)=2\cdot\frac{1}{2^n}=\frac{1}{2^{n-1}}.$$

2. We proceed in three steps. First, we simplify the problem, based on its symmetry. Second, we state strict dominance implications that allow us to formulate a recursive problem. Finally, we solve this problem by connecting it to the recurrence relation of the unsigned Stirling numbers of the first kind.

Step 1. Reformulation

Because of symmetry, we can normalize the first Column's payoff row $c_{1\cdot}$ to any fixed permutation, e.g. to the identity permutation $e_n \equiv (1,2,\ldots,n)$. That is,

$$C=\begin{pmatrix} 1 & 2 & \ldots & j & \ldots & n \\ c_{21} & c_{22} & \ldots & c_{2j} & \ldots & c_{2n} \end{pmatrix}.$$

This is without loss of generality. Indeed, for any realization of C, we can multiply its rows by the same $\sigma \in S_n$ to get an equivalent realization (up to relabeling) with respect to the strict dominance as a strict (of irreflexive) partial order relation.

Step 2. Strict Dominance Implications

For this simplified problem, we have the following observations:

(a) If $c_{2j}=n$ for some j, then the j-th column is strictly undominated.

(b) If $c_{2j}=n$ for some j, then the j-th column strictly dominates columns $i \leq j-1$.

(c) For any j, the j-th column does not strictly dominate columns $i \geq j+1$.

Step 3. Recursive Problem

Let $f(n,k)$, $k\in[n]$, denote the number of permutations $c_{2\cdot}\in S_n$ such that Column has exactly k strictly undominated actions, i.e. $f(n,k)=n!\cdot\Pr\left(U^C(2,n)=k\right)$.

There are two possible cases. First, if $\sigma(1)=n$, then the 1-st column is strictly undominated and we need to guarantee $k-1\geq 0$ strictly undominated actions out of remaining

$n-1$ actions. Second, if $\sigma(1) < n$, then the 1-st column is strictly dominated and we need to guarantee $k \geq 1$ strictly undominated actions out of remaining $n-1$ actions. Therefore,

$$f(n,k) = f(n-1,k-1) + (n-1)f(n-1,k),$$

and $f(n,k)$ satisfies the standard recurrence for the unsigned Stirling numbers of the first kind with appropriate initial conditions (e.g., 1.3.6 Lemma in Stanley, 2011). $\square$

Proof of Proposition 1. Let's prove three statements in turn.

1. By Lemma 1,

$$\begin{aligned}\pi(2,n) = \Pr(S^R(2,n)=1) &= \sum_{k=1}^{n} \Pr\left(U^C(2,n)=k\right)\cdot \Pr\left(S^R(2,n)=1 \mid U^C(2,n)=k\right)\\ &= \sum_{k=1}^{n} \Pr\left(U^C(2,n)=k\right)\cdot \Pr\left(U^R(2,k)=1\right) = \frac{2}{n!}\cdot\left.\left(\sum_{k=1}^{n} c(n,k)\cdot x^k\right)\right|_{x=1/2},\end{aligned}$$

where the third equality follows from the ordinal randomness assumption. By using the Pochhammer symbol $x^{(n)} \equiv x(x+1)\dots(x+n-1)$, we get

$$\pi(2,n) = \frac{2}{n!}\cdot x^{(n)}\big|_{x=1/2} = \frac{2}{\Gamma(n+1)}\cdot\frac{\Gamma(n+1/2)}{\Gamma(1/2)},$$

where $\Gamma(\cdot)$ is the gamma function with $\Gamma(1/2) = \sqrt{\pi}$, the first equality follows from the Proposition 1.3.7 in Stanley (2015), and the second equality is standard (e.g., Srivastava, 2013). By introducing the Wallis ratio

$$W(n) \equiv \frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma\left(n+1/2\right)}{\Gamma(1/2)\Gamma(n+1)},$$

we finally obtain $\pi(2,n) = 2W(n) = \dfrac{(2n-1)!!}{2^{n-1}\cdot n!}$.

2. For any $n \geq 1$, by using the identity $\Gamma(x+1) = x\Gamma(x)$,

$$\pi(2,n+1) = \frac{n+1/2}{n+1}\cdot\pi(2,n) < \pi(2,n).$$

3. By Stirling's formula applied to the gamma function,

$$\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)\cdot n^{\alpha}} = 1,$$

so that

$$\lim_{n\to\infty} n^{1/2}\cdot \pi(2,n) = \frac{2}{\sqrt{\pi}}\cdot \lim_{n\to\infty} \frac{\Gamma(n+1/2)\cdot n^{1/2}}{\Gamma(n+1)} = \frac{2}{\sqrt{\pi}}. \qquad \square$$

Lemma 4. *Consider a random game $G(2,n)$. Then,*

1. $\Pr(I(2,n)=1)$ *is strictly decreasing in n and* $\lim_{n\to\infty} 2^n\cdot n^{1/2}\cdot \Pr(I(2,n)=1) = \sqrt{\pi}$;
2. $\Pr(I(2,1)=2)=0$, $\Pr(I(2,2)=2)=\Pr(I(2,3)=2)=2/3$, $\Pr(I(2,n)=2)$ *is strictly decreasing in n for $n\geq 3$, and* $\lim_{n\to\infty} n^{1/2}\cdot \Pr(I(2,n)=2) = \frac{\sqrt{\pi}}{2}$;
3. $\Pr(I(2,1)=3)=\Pr(I(2,2)=3)=0$, $\Pr(I(2,n)=3)$ *is strictly increasing in n for $n\geq 2$, and* $\lim_{n\to\infty} n^{1/2}\cdot (1-\Pr(I(2,n)=3)) = \frac{\sqrt{\pi}}{2}$.

Proof. Let's prove three statements in turn.

1. Consider $\Pr(I(2,n)=1) = \frac{\sqrt{\pi}}{2^n}\cdot\frac{\Gamma(n)}{\Gamma(n+1/2)}$ (derived in Subsection 3.3). For any $n\geq 1$,

$$\Pr(I(2,n+1)=1) = \frac{\sqrt{\pi}}{2^{n+1}}\cdot \frac{\Gamma(n+1)}{\Gamma(n+3/2)} = \frac{n}{2n+1}\cdot \Pr(I(2,n)=1) < \Pr(I(2,n)=1).$$

By Stirling's formula applied to the gamma function,

$$\lim_{n\to\infty} 2^n\cdot n^{1/2}\cdot \Pr(I(2,n)=1) = \sqrt{\pi}\lim_{n\to\infty}\frac{\Gamma(n)\cdot n^{1/2}}{\Gamma(n+1)} = \sqrt{\pi}.$$

2. Then, consider $\Pr(I(2,n)=2) = \frac{n+2^{n-1}-2}{2^n}\cdot\sqrt{\pi}\cdot\frac{\Gamma(n)}{\Gamma(n+1/2)}$ (derived in Subsection 3.3). It is trivial to check that $\Pr(I(2,2)=2)=\Pr(I(2,3)=2)=2/3$ and $\Pr(I(2,1)=2)=0$. For any $n\geq 3$,

$$\Pr(I(2,n+1)=2) = \frac{n+2^n-1}{2n+2^n-4}\cdot\frac{n}{n+1/2}\cdot \Pr(I(2,n)=2) < \Pr(I(2,n)=2).$$

By Stirling's formula applied to the gamma function,

$$\lim_{n\to\infty} n^{1/2}\cdot \Pr(I(2,n)=2) = \frac{\sqrt{\pi}}{2}\cdot \lim_{n\to\infty}\frac{n+2^{n-1}-2}{2^{n-1}}\cdot \lim_{n\to\infty}\frac{\Gamma(n)\cdot n^{1/2}}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2}.$$

3. Finally, consider $\Pr(I(2,n)=3) = 1-\dfrac{n+2^{n-1}-1}{2^n}\cdot\sqrt{\pi}\cdot\dfrac{\Gamma(n)}{\Gamma(n+1/2)}$ (derived in Subsection 3.3). It is trivial to check that $\Pr(I(2,1)=3)=\Pr(I(2,2)=3)=0$. By 1 and 2, $\Pr(I(2,n)=3)$ is strictly increasing in n for $n\geq 2$. By Stirling's formula,

$$\lim_{n\to\infty} n^{1/2}\cdot (1-\Pr(I(2,n)=3)) = \frac{\sqrt{\pi}}{2}\cdot \lim_{n\to\infty}\frac{n+2^{n-1}-1}{2^{n-1}}\cdot \lim_{n\to\infty}\frac{\Gamma(n)\cdot n^{1/2}}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2}. \qquad \square$$

Proof of Proposition 2. Let's prove three statements in turn.

1. By using probabilities derived in Subsection 3.3 and stated in Lemma 4,

$$\begin{aligned}\mathbb{E}\left[I(2,n)\right] = \sum_{i=1}^{3} i\cdot \Pr(I(2,n)=i) &= 1\cdot\frac{\sqrt{\pi}}{2^n}\cdot\frac{\Gamma(n)}{\Gamma(n+1/2)} + 2\cdot\frac{n+2^{n-1}-2}{2^n}\cdot\sqrt{\pi}\cdot\frac{\Gamma(n)}{\Gamma(n+1/2)}\\ &+3\cdot\left(1-\frac{n+2^{n-1}-1}{2^n}\cdot\sqrt{\pi}\cdot\frac{\Gamma(n)}{\Gamma(n+1/2)}\right) = 3-\frac{n+2^{n-1}}{2^n}\cdot\sqrt{\pi}\cdot\frac{\Gamma(n)}{\Gamma(n+1/2)}.\end{aligned}$$

2. Note that for any $n\geq 1$,

$$A(n+1) \equiv \frac{n+1+2^n}{2^{n+1}}\cdot\sqrt{\pi}\cdot\frac{\Gamma(n+1)}{\Gamma(n+3/2)} = \frac{n+1+2^n}{2n+2^n}\cdot\frac{n}{n+1/2}\cdot A(n) < A(n),$$

so that $\mathbb{E}\left[I(2,n)\right]$ is strictly increasing in n by 1.

3. Similar to the proof of Lemma 4, we have $\lim_{n\to\infty} n^{1/2}\cdot(3-\mathbb{E}\left[I(2,n)\right]) = \dfrac{\sqrt{\pi}}{2}$. $\square$

Lemma 5. *Consider a random game $G(2,n)$. Then,*

1. $\displaystyle\lim_{n\to\infty} n^{1/2}\cdot\Pr\left(S^C(2,n)=1\right) = \frac{2}{\sqrt{\pi}}$;
2. *for any fixed $k\geq 2$,* $\displaystyle\lim_{n\to\infty}\frac{n}{(\log n)^{k-1}}\cdot\Pr\left(S^C(2,n)=k\right) = \frac{1}{(k-1)!}\cdot\left(1-\frac{1}{2^{k-1}}\right)$;
3. *for $k(n)\sim\log n$,* $\displaystyle\lim_{n\to\infty}(\log n)^{1/2}\cdot\Pr\left(S^C(2,n)=k(n)\right) = \frac{1}{\sqrt{2\pi}}$.

Proof. For these results, we use two nontrivial theorems related to the unsigned Stirling numbers of the first kind.

Hwang's Theorem (Theorem 1 for $\nu = 0$ in Hwang, 1995)**.** For any $\eta > 0$, the unsigned Stirling numbers of the first kind $c(n,k)$ satisfy asymptotically

$$\frac{c(n,k)}{n!} = \frac{1}{n} \cdot \frac{(\log n + \gamma)^{k-1}}{(k-1)!} + \mathcal{O}\left(\frac{(\log n)^k}{k! \cdot n^2}\right) \quad (n \to \infty),$$

uniformly for $1 \leq k \leq \eta \log n$.

Erdős Theorem (e.g., see p.124 in Stanley, 2011)**.** The Stirling numbers of the first kind form log-concave sequences. In addition, the signless Stirling number $c(n,k)$ is maximized at $k(n) = \arg\max_{k \in \{\lfloor H_n \rfloor, \lceil H_n \rceil\}} c(k,n)$, where $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the n-th harmonic number. That is, $k(n) \sim \log n$.

We now prove three statements of the lemma in sequence.

1. It follows immediately from Proposition 1.

2. It follows immediately from Hwang's theorem for fixed $k \geq 2$. Actually, the original theorem for this case belongs to Wilf (1993).

3. By Hwang's theorem,

$$\frac{c(n,k(n))}{n!} \sim \frac{1}{n} \cdot \frac{(\log n)^{\log n - 1}}{\Gamma(\log n)} \quad (n \to \infty,\ k(n) \sim \log n). \tag{†}$$

By applying Stirling's formula for the gamma function

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)$$

to $\Gamma(\log n)$, we get

$$\Gamma(\log n) \sim \frac{\sqrt{2\pi}}{n} \cdot (\log n)^{\log n - \frac{1}{2}} \quad (n \to \infty), \tag{††}$$

so that from equations † and †† we have

$$\frac{c(n,k(n))}{n!} \sim \frac{1}{\sqrt{2\pi}} \cdot (\log n)^{-\frac{1}{2}} \quad (n \to \infty,\ j(n) \sim \log n).$$

Thus,

$$\Pr\left(S^C(2,n)=k(n)\right)=\frac{c(n,k(n))}{n!}\cdot\left(1-\frac{1}{2^{k(n)-1}}\right)\sim\frac{1}{\sqrt{2\pi}}\cdot(\log n)^{-\frac{1}{2}}\quad(n\to\infty,\ k(n)\sim\log n)$$

and $\lim_{n\to\infty}(\log n)^{1/2}\cdot\Pr\left(S^C(2,n)=k(n)\right)=\frac{1}{\sqrt{2\pi}}$ as desired. □

Lemma 6. *Consider a random game $G(2,n)$. Then,*

1. *for any $n\geq 1$, $\mathbb{E}\left[\left|S^C(2,n)\right|\right]=W(n)\cdot\left(2-(\psi(n+1/2)-\psi(1/2))\right)+H_n$, where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function and $W(n)=\frac{\Gamma(n+1/2)}{\Gamma(n+1)\cdot\Gamma(1/2)}$;*
2. *$\mathbb{E}\left[\left|S^C(2,n)\right|\right]$ is strictly increasing in n;*
3. *$\mathbb{E}\left[\left|S^C(2,n)\right|\right]=\log n+\gamma-\frac{1}{\sqrt{\pi}}\cdot\frac{\log n}{n^{1/2}}+\mathcal{O}\left(\frac{1}{n^{1/2}}\right)$, where γ is the Euler-Mascheroni constant.*

Proof. Let's prove three statements in turn.

1. Recall that

$$\Pr\left(\left|S^C(2,n)\right|=1\right)=\frac{1}{n!}\cdot\sum_{k=1}^{n}\frac{c(n,k)}{2^{k-1}},\text{ and}$$
$$\Pr\left(\left|S^C(2,n)\right|=k\right)=\frac{c(n,k)}{n!}\cdot\left(1-\frac{1}{2^{k-1}}\right)\quad\text{for any }2\leq k\leq n,$$

so that

$$\mathbb{E}\left[\left|S^C(2,n)\right|\right]=\frac{1}{n!}\cdot\sum_{k=1}^{n}\frac{c(n,k)}{2^{k-1}}+\sum_{k=2}^{n}k\cdot\frac{c(n,k)}{n!}\cdot\left(1-\frac{1}{2^{k-1}}\right)=\frac{1}{n!}\cdot\sum_{k=1}^{n}\frac{c(n,k)}{2^{k-1}}$$
$$+\sum_{k=1}^{n}k\cdot\frac{c(n,k)}{n!}\cdot\left(1-\frac{1}{2^{k-1}}\right)=2W(n)+\sum_{k=1}^{n}k\cdot\frac{c(n,k)}{n!}-\sum_{k=1}^{n}k\cdot\frac{c(n,k)}{2^{k-1}\cdot n!}.$$

First, note that (e.g., Theorem 2 in Benjamin et al., 2002) $\sum_{k=1}^{n}k\cdot\frac{c(n,k)}{n!}=H_n$.

Second, let's differentiate the identity $\sum_{k=1}^{n} c(n,k) \cdot x^k = \frac{\Gamma(n+x)}{\Gamma(x)}$ to derive the explicit expression for $\sum_{k=1}^{n} k \cdot \frac{c(n,k)}{2^{k-1} \cdot n!}$. That is,

$$\sum_{k=1}^{n} k \cdot c(n,k) \cdot x^{k-1} = \frac{d}{dx}\left(\frac{\Gamma(n+x)}{\Gamma(x)}\right) = \frac{\Gamma(n+x)}{\Gamma(x)} \cdot (\psi(n+x) - \psi(x)),$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, so that $\sum_{k=1}^{n} k \cdot \frac{c(n,k)}{2^{k-1} \cdot n!} = W(n) \cdot (\psi(n+1/2) - \psi(1/2))$.

By collecting all terms, we get

$$\mathbb{E}\left[\left|S^C(2,n)\right|\right] = W(n) \cdot \big(2 - (\psi(n+1/2) - \psi(1/2))\big) + H_n,$$

where $\psi(n+1/2) = -\gamma - 2\log 2 + \sum_{k=1}^{n} \frac{2}{2k-1} = -\gamma + H_{n-1/2}$ and $\psi(1/2) = -\gamma - 2\log 2$.

2. Note that for any $n \geq 1$,

$$\begin{aligned}\mathbb{E}\left[\left|S^C(2,n+1)\right|\right] &= W(n+1) \cdot \big(2 - (\psi(n+3/2) - \psi(1/2)\big) + H_{n+1} \text{ and}\\ \mathbb{E}\left[\left|S^C(2,n)\right|\right] &= W(n) \cdot \big(2 - (\psi(n+1/2) - \psi(1/2)\big) + H_n,\end{aligned}$$

so that

$$\begin{aligned}&\mathbb{E}\left[\left|S^C(2,n+1)\right|\right] - \mathbb{E}\left[\left|S^C(2,n)\right|\right] = \frac{1}{n+1}\\ &+ W(n) \times \left(-\frac{1}{n+1} + (\psi(n+1/2) - \psi(1/2)) - \frac{n+1/2}{n+1} \cdot \big(\psi(n+3/2) - \psi(1/2)\big)\right)\\ &> W(n) \cdot \left(\sum_{k=1}^{n} \frac{2}{2k-1} - \frac{n+1/2}{n+1} \cdot \sum_{k=1}^{n+1} \frac{2}{2k-1}\right) = \frac{W(n)}{n+1} \cdot \left(\sum_{k=1}^{n} \frac{1}{2k-1} - 1\right) \geq 0,\end{aligned}$$

where the first inequality follows from $W(n) = \frac{\pi(2,n)}{2} < 1$ for any $n \geq 1$.

3. Note that

$$H_n = \log n + \gamma + \mathcal{O}\left(\frac{1}{n}\right), \qquad \frac{\Gamma(n+1/2)}{\Gamma(n+1)} = \frac{1}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right), \qquad \text{and}$$
$$\psi(n+1/2) = \log(n+1/2) + \mathcal{O}\left(\frac{1}{n}\right),$$

so that

$$\mathbb{E}\left[\left|S^C(2,n)\right|\right] = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{1}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right) \cdot \left(2 + \gamma + 2\log 2 - \log(n+1/2) - \mathcal{O}\left(\frac{1}{n}\right)\right)$$
$$+ \log n + \gamma + \mathcal{O}\left(\frac{1}{n}\right) = \log n + \gamma - \frac{1}{\sqrt{\pi}} \cdot \frac{\log n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right).$$

In particular, $\lim_{n\to\infty} \left(\mathbb{E}\left[\left|S^C(2,n)\right|\right] - \log n\right) = \gamma.$ □

Lemma 7. *Consider a random game $G(2,n)$. Let's define the polygamma function $\psi^{(m)}(x)$ of order m as the m-th derivative of the digamma function $\psi(x) \equiv \psi^{(0)}(x)$. In addition, $H_n^{(m)} \equiv 1 + \frac{1}{2^m} + \ldots + \frac{1}{n^m}$ is the generalized harmonic number of order m of n. Then,*

1. *for any $n \geq 1$,*

$$\operatorname{Var}\left[S^C(2,n)\right] = H_n - H_n^{(2)} + \frac{W(n)}{2} \cdot \Bigg(4 - 2 \cdot \left(\psi(n+1/2) - \psi(1/2)\right)$$
$$-\left(\left(\psi(n+1/2)-\psi(1/2)\right)^2 + \left(\psi^{(1)}(n+1/2) - \psi^{(1)}(1/2)\right)\right) - 8 \cdot H_n + 4 \cdot H_n \cdot \left(\psi(n+1/2) - \psi(1/2)\right)\Bigg)$$
$$- \left(W(n) \cdot \left(2 - \left(\psi(n+1/2) - \psi(1/2)\right)\right)\right)^2.$$

2. $\operatorname{Var}\left[S^C(2,n)\right] = \log n + \gamma - \frac{\pi^2}{6} + \frac{3}{2 \cdot \sqrt{\pi}} \cdot \frac{(\log n)^2}{n^{1/2}} - \frac{5 - 2\log 2 - 3\gamma}{\sqrt{\pi}} \cdot \frac{\log n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right).$

Proof. Let's prove two statements in turn.

1. Similar to the proof of Lemma 6,

$$\mathbb{E}\left[S^C(2,n)^2\right] = \frac{1}{n!} \cdot \sum_{j=1}^{n} \frac{c(n,j)}{2^{j-1}} + \sum_{j=2}^{n} j^2 \cdot \frac{c(n,j)}{n!} \cdot \left(1 - \frac{1}{2^{j-1}}\right) = \frac{1}{n!} \cdot \sum_{j=1}^{n} \frac{c(n,j)}{2^{j-1}}$$

$$+\sum_{j=1}^{n} j^2 \cdot \frac{c(n,j)}{n!} \cdot \left(1 - \frac{1}{2^{j-1}}\right) = 2W(n) + \sum_{j=1}^{n} j^2 \cdot \frac{c(n,j)}{n!} - \sum_{j=1}^{n} j^2 \cdot \frac{c(n,j)}{2^{j-1} \cdot n!}$$
$$= 2W(n) + \sum_{j=1}^{n} j^2 \cdot \frac{c(n,j)}{n!} - \sum_{j=1}^{n} j \cdot \frac{c(n,j)}{2^{j-1} \cdot n!} - \frac{1}{2} \cdot \sum_{j=1}^{n} j(j-1) \cdot \frac{c(n,j)}{2^{j-2} \cdot n!}.$$

First, we have $\sum_{j=1}^{n} j^2 \cdot \frac{c(n,j)}{n!} = H_n + H_n^2 - H_n^{(2)}$ (Gontcharoff, 1944). Second, note that $\sum_{j=1}^{n} j \cdot \frac{c(n,j)}{2^{j-1} \cdot n!} = W(n) \cdot (\psi(n+1/2) - \psi(1/2))$.

Third, let's differentiate twice the identity $\sum_{j=1}^{n} c(n,j) \cdot x^j = \frac{\Gamma(n+x)}{\Gamma(x)}$ to derive the explicit expression for $\sum_{j=1}^{n} j(j-1) \cdot \frac{c(n,j)}{2^{j-2} \cdot n!}$. That is,

$$\sum_{j=1}^{n} j(j-1) \cdot c(n,j) \cdot x^{j-2} = \frac{d^2}{dx^2}\left(\frac{\Gamma(n+x)}{\Gamma(x)}\right) = \frac{d}{dx}\left(\frac{\Gamma(n+x)}{\Gamma(x)} \cdot (\psi(n+x) - \psi(x))\right)$$
$$= \frac{\Gamma(n+x)}{\Gamma(x)} \cdot \left(\big(\psi(n+x) - \psi(x)\big)^2 + \big(\psi^{(1)}(n+x) - \psi^{(1)}(x)\big)\right),$$

where $\psi^{(m)}(z) = \frac{d^m}{dz^m}\psi(z)$ the polygamma function of order m, so that

$$\sum_{j=1}^{n} j(j-1) \cdot \frac{c(n,j)}{2^{j-2} \cdot n!} = W(n) \cdot \left(\big(\psi(n+1/2) - \psi(1/2)\big)^2 + \big(\psi^{(1)}(n+1/2) - \psi^{(1)}(1/2)\big)\right).$$

Thus,

$$\mathbb{E}\left[S^C(2,n)^2\right] = 2W(n) + \left(H_n + H_n^2 - H_n^{(2)}\right) - W(n) \cdot (\psi(n+1/2) - \psi(1/2))$$
$$- \frac{W(n)}{2} \cdot \left(\big(\psi(n+1/2) - \psi(1/2)\big)^2 + \big(\psi^{(1)}(n+1/2) - \psi^{(1)}(1/2)\big)\right).$$

By Lemma 6,

$$\mathbb{E}\left[S^C(2,n)\right] = W(n) \cdot \big(2 - (\psi(n+1/2) - \psi(1/2))\big) + H_n,$$

so that by using $\operatorname{Var}\left[S^C(2,n)\right] = \mathbb{E}\left[S^C(2,n)^2\right] - \mathbb{E}\left[S^C(2,n)\right]^2$, we get the desired expression.

2. By using asymptotic expressions for each term, we get

$$\begin{aligned}\operatorname{Var}\left[S^C(2,n)\right] &= \log n + \gamma - \frac{\pi^2}{6} + \frac{1}{2\cdot\sqrt{\pi}}\cdot\left(\frac{1}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right)\\ &\times\Bigg(-(\log(n+1/2))^2 + 4\cdot\log(n+1/2)\cdot\log n - 2\log(n+1/2)\\ &-(4\log 2+2\gamma)\log(n+1/2) - 8\log n + 4\gamma\cdot\log n\\ &+(8\log 2+4\gamma)\log(n+1/2) + \mathcal{O}(1)\Bigg)\\ &= \log n + \gamma - \frac{\pi^2}{6} + \frac{3}{2\cdot\sqrt{\pi}}\cdot\frac{(\log n)^2}{n^{1/2}} - \frac{5-2\log 2-3\gamma}{\sqrt{\pi}}\cdot\frac{\log n}{n^{1/2}} + \mathcal{O}\left(\frac{1}{n^{1/2}}\right),\end{aligned}$$

where $\lim_{n\to\infty} H_n^{(2)} = \dfrac{\pi^2}{6}$. □

Proof of Proposition 3. The proof of this statement is similar to Hwang (1998) and uses the Berry-Esseen theorem to find the convergence rate in the stated central limit result. The difference is that the problem does not belong to the exp-log class immediately (one can observe several exp-terms below after some calculations), but still the same reasoning can be applied to establish the result.

Berry–Esseen theorem (Theorem 2 in Petrov, 1975)**.** Let $F(x)$ be a non-decreasing function, $G(x)$ a differentiable function of bounded variation on the real line, $\varphi(t)$ and $\gamma(t)$ the corresponding Fourier-Stieltjes transforms:

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx}dF(x), \quad \gamma(t) = \int_{-\infty}^{\infty} e^{itx}dG(x).$$

Suppose that $F(-\infty) = G(-\infty)$, $F(\infty) = G(\infty)$, T is an arbitrary positive number, and $|G'(x)| \le A$. Then for every $b > 1/(2\pi)$ we have

$$\sup_{-\infty<x<\infty} |F(x) - G(x)| \le b\int_{-T}^{T}\left|\frac{\varphi(t)-\gamma(t)}{t}\right| dt + r(b)\frac{A}{T},$$

where $r(b)$ is a positive constant depending only on b.

We proceed in two steps. In step 1, we reformulate the problem by using the Berry-Esseen inequality. In step 2, we calculate the characteristic function and use it to establish the result.

Step 1. Reformulated problem

Let's take $G(x) = \Phi(x)$ (so that $A = 1/\sqrt{2\pi}$) and $T = T_n = c\sigma_n$, where $c > 0$ is a sufficiently small constant. By the Berry-Esseen inequality, it will be sufficient to prove that

$$J_n = \int_{-T_n}^{T_n} \left| \frac{\varphi_n(t) - e^{-\frac{1}{2}t^2}}{t} \right| dt = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right).$$

Step 2. Characteristic function

Let $\varphi_n(t)$ denote the characteristic function of $\dfrac{S^C(2,n) - e_n}{\sigma_n}$, the normed random variable, where by Lemmas 6 and 7,

$$\begin{aligned}
e_n &\equiv \mathbb{E}\left[S^C(2,n)\right] = \log n + \gamma + o(1), \text{ and} \\
\sigma_n &\equiv \sqrt{\operatorname{Var}\left[S^C(2,n)\right]} = \sqrt{\log n} - \left(\frac{\pi^2}{12} - \frac{\gamma}{2}\right) \cdot \frac{1}{\sqrt{\log n}} + o\left(\frac{1}{\sqrt{\log n}}\right).
\end{aligned}$$

That is,

$$\begin{aligned}
\varphi_n(t) &= \sum_{j=1}^{n} \Pr\left(S^C(2,n) = j\right) \cdot e^{it(j-e_n)/\sigma_n} \\
&= e^{-ite_n/\sigma_n} \cdot \left(\frac{1}{n!} \cdot \sum_{j=1}^{n} \frac{c(n,j)}{2^{j-1}} \cdot e^{it/\sigma_n} + \sum_{j=2}^{n} \frac{c(n,j)}{n!} \cdot \left(1 - \frac{1}{2^{j-1}}\right) \cdot e^{itj/\sigma_n}\right) \\
&= e^{-ite_n/\sigma_n} \cdot \left(\frac{1}{n!} \cdot \sum_{j=1}^{n} \frac{c(n,j)}{2^{j-1}} \cdot e^{it/\sigma_n} + \sum_{j=1}^{n} \frac{c(n,j)}{n!} \cdot \left(1 - \frac{1}{2^{j-1}}\right) \cdot e^{itj/\sigma_n}\right) \\
&= e^{-ite_n/\sigma_n} \cdot \left(\sum_{j=1}^{n} \frac{c(n,j)}{n!} \cdot e^{itj/\sigma_n}\right) \\
&+ 2 \cdot e^{-ite_n/\sigma_n} \cdot \left(\frac{1}{n!} \cdot \sum_{j=1}^{n} \frac{c(n,j)}{2^j} \cdot e^{it/\sigma_n} - \sum_{j=1}^{n} \frac{c(n,j)}{n!} \cdot \frac{1}{2^j} \cdot e^{itj/\sigma_n}\right) = A_n(t) + B_n(t),
\end{aligned}$$

where

$$\begin{aligned}
A_n(t) &\equiv e^{-ite_n/\sigma_n} \cdot \frac{1}{\Gamma\left(e^{it/\sigma_n}\right)} \cdot \frac{\Gamma\left(n + e^{it/\sigma_n}\right)}{\Gamma(n+1)}, \text{ and} \\
B_n(t) &\equiv 2 \cdot e^{-ite_n/\sigma_n} \cdot \left(\frac{e^{it/\sigma_n}}{\Gamma(1/2)} \cdot \frac{\Gamma(n+1/2)}{\Gamma(n+1)} - \frac{1}{\Gamma\left(e^{it/\sigma_n}/2\right)} \cdot \frac{\Gamma\left(n + e^{it/\sigma_n}/2\right)}{\Gamma(n+1)}\right).
\end{aligned}$$

First, let's find the asymptotic expression for $A_n(t)$. By denoting $e^{it/\sigma_n} \equiv 1 + \varepsilon_n$ with $\varepsilon_n = \dfrac{it}{\sigma_n} - \dfrac{t^2}{2\sigma_n^2} + \mathcal{O}\left(\dfrac{|t|^3}{\sigma_n^3}\right)$ and using Stirling's formula,

$$\log \frac{\Gamma\left(n + e^{it/\sigma_n}\right)}{\Gamma(n+1)} = \log \Gamma\left(n + 1 + \varepsilon_n\right) - \log \Gamma\left(n+1\right)$$
$$= (n+1/2+\varepsilon_n)\cdot\log\left(n+1+\varepsilon_n\right) - (n+1+\varepsilon_n) + \frac{1}{2}\log 2\pi - (n+1/2)\cdot\log\left(n+1\right) + (n+1) - \frac{1}{2}\log 2\pi + \mathcal{O}\left(\frac{1}{n}\right)$$
$$= \varepsilon_n \log(n+1+\varepsilon_n) + (n+1/2)\log\left(1 + \frac{\varepsilon_n}{n+1}\right) - \varepsilon_n + \mathcal{O}\left(\frac{1}{n}\right) = \varepsilon_n \log n + \mathcal{O}\left(\frac{1}{n}\right)$$
$$= \left(\frac{it}{\sigma_n} - \frac{t^2}{2\sigma_n^2} + \mathcal{O}\left(\frac{|t|^3}{\sigma_n^3}\right)\right)\cdot \log n + \mathcal{O}\left(\frac{1}{n}\right) = \left(\frac{it}{\sqrt{\log n}} - \frac{t^2}{2\log n} + \mathcal{O}\left(\frac{|t|^3}{(\log n)^{3/2}}\right)\right)\cdot \log n + \mathcal{O}\left(\frac{1}{n}\right)$$
$$= it \cdot \sqrt{\log n} - \frac{t^2}{2} + \mathcal{O}\left(\frac{|t|^3}{\sqrt{\log n}}\right).$$

In addition,

$$\log \Gamma\left(e^{it/\sigma_n}\right) = \log \Gamma\left(1 + \mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)\right) = \mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right) \qquad \text{and}$$
$$\frac{ite_n}{\sigma_n} = it \times \frac{\log n + \gamma + o(1)}{\sqrt{\log n} - \left(\frac{\pi^2}{12} - \frac{\gamma}{2}\right)\cdot \frac{1}{\sqrt{\log n}} + o\left(\frac{1}{\sqrt{\log n}}\right)} = it \cdot \sqrt{\log n} + \mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right).$$

By collecting all terms, we get

$$A_n(t) = e^{-\left(it\cdot\sqrt{\log n} + \mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)\right)} \cdot e^{-\mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)} \cdot e^{it\cdot\sqrt{\log n} - \frac{t^2}{2} + \mathcal{O}\left(\frac{|t|^3}{\sqrt{\log n}}\right)} = e^{-\frac{t^2}{2} + \mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\log n}}\right)}.$$

Second, let's find the asymptotic expression for $B_n(t)$. By using similar calculations,

$$\log \frac{\Gamma\left(n + 1/2\right)}{\Gamma(n+1)} = \log \Gamma\left(n + 1/2\right) - \log \Gamma\left(n+1\right)$$
$$= n\cdot\log\left(n+1/2\right) - (n+1/2) + \frac{1}{2}\log 2\pi - (n+1/2)\cdot\log\left(n+1\right) + (n+1) - \frac{1}{2}\log 2\pi + \mathcal{O}\left(\frac{1}{n}\right)$$
$$= -\frac{1}{2}\cdot\log\left(n+1\right) + n\log\left(1 - \frac{1}{2}\cdot\frac{1}{n+1}\right) - 1/2 + \mathcal{O}\left(\frac{1}{n}\right) = -\frac{1}{2}\cdot\log n + \mathcal{O}\left(\frac{1}{n}\right),$$

and

$$\log \frac{\Gamma\left(n+e^{it/\sigma_n}/2\right)}{\Gamma(n+1)} = \log\Gamma\left(n+1/2+\varepsilon_n/2\right) - \log\Gamma\left(n+1\right)$$
$$= \left(n+\varepsilon_n/2\right)\cdot\log\left(n+1/2+\varepsilon_n/2\right) - \left(n+1/2+\varepsilon_n/2\right) + \frac{1}{2}\log 2\pi$$
$$- \left(n+1/2\right)\cdot\log\left(n+1\right) + \left(n+1\right) - \frac{1}{2}\log 2\pi + \mathcal{O}\left(\frac{1}{n}\right)$$
$$= \left(\varepsilon_n/2-1/2\right)\log n + \left(n+\varepsilon_n/2\right)\log\left(1+\frac{\varepsilon_n/2-1/2}{n+1/2+\varepsilon_n/2}\right) + \left(1/2-\varepsilon_n/2\right) + \mathcal{O}\left(\frac{1}{n}\right)$$
$$= -\frac{1}{2}\log n + \frac{1}{2}it\cdot\sqrt{\log n} + \mathcal{O}\left(t^2\right).$$

Furthermore,

$$\log\Gamma\left(e^{it/\sigma_n}/2\right) = \log\Gamma\left(1/2+\mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)\right) = \mathcal{O}\left(1\right) \qquad \text{and}$$
$$\frac{it}{\sigma_n} = it\cdot\frac{1}{\sqrt{\log n} - \left(\frac{\pi^2}{12}-\frac{\gamma}{2}\right)\cdot\frac{1}{\sqrt{\log n}} + o\left(\frac{1}{\sqrt{\log n}}\right)} = \mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right).$$

By collecting all terms, we get

$$B_n(t) = \frac{2}{\sqrt{\pi}}\cdot e^{-\left(it\cdot\sqrt{\log n}+\mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)\right)}\cdot e^{\mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)}\cdot e^{-\frac{1}{2}\cdot\log n+\mathcal{O}\left(\frac{1}{n}\right)}$$
$$-2\cdot e^{-\left(it\cdot\sqrt{\log n}+\mathcal{O}\left(\frac{|t|}{\sqrt{\log n}}\right)\right)}\cdot e^{-\mathcal{O}(1)}\cdot e^{-\frac{1}{2}\cdot\log n+\frac{1}{2}it\cdot\sqrt{\log n}+\mathcal{O}\left(t^2\right)},$$

i.e. $B_n(s) = \mathcal{O}\left(\frac{e^{\tau\cdot\sqrt{\log n}}}{n^{1/2}}\right)$ uniformly for $|s|\le\tau$, $s\in\mathcal{C}$ for some fixed $\tau>0$. Let $\kappa_n \equiv \frac{n^{1/2}}{e^{\tau\cdot\sqrt{\log n}}}$ for simplicity.

Thus,

$$\varphi_n(t) = A_n(t) + B_n(t) = e^{-\frac{t^2}{2}+\mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\log n}}\right)} + \mathcal{O}\left(\frac{1}{\kappa_n}\right).$$

In addition, by taking a small circle around the origin we easily obtain $B_n(s) = \mathcal{O}\left(\frac{|s|}{\kappa_n}\right)$ for $|s|\le c<\tau$, so that

$$\varphi_n(t) = e^{-\frac{t^2}{2}+\mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\log n}}\right)} + \mathcal{O}\left(\frac{|t|}{\kappa_n\cdot\sqrt{\log n}}\right),$$

for $|t|\le T_n$.

Note that we can use Levy's convergence theorem to obtain the convergence result. However, we still need to use the Berry-Esseen inequality to find the convergence rate.

Based on the obtained approximation, we can follow the proof by Hwang (1998). That is, using the inequality $|e^w - 1| \le |w|e^{|w|}$ for all complex w, we obtain

$$\begin{aligned}\left|\frac{\varphi_n(t) - e^{-\frac{1}{2}t^2}}{t}\right| &= \mathcal{O}\left(\left(\frac{1+t^2}{\sqrt{\log n}}\right)\exp\left(-\frac{t^2}{2} + \mathcal{O}\left(\frac{|t|+|t|^3}{\sqrt{\log n}}\right)\right) + \frac{1}{\kappa_n \cdot \sqrt{\log n}}\right) \\ &= \mathcal{O}\left(\left(\frac{1+t^2}{\sqrt{\log n}}\right)e^{-\frac{1}{4}t^2} + \frac{1}{\kappa_n \cdot \sqrt{\log n}}\right) \quad (|t| \le T_n),\end{aligned}$$

for sufficiently small $0 < c < \tau$. Thus,

$$\begin{aligned}J_n &= \int_{-T_n}^{T_n}\left|\frac{\varphi_n(t) - e^{-\frac{1}{2}t^2}}{t}\right| dt = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\int_{-T_n}^{T_n}\left(1+t^2\right)e^{-\frac{1}{4}t^2}dt + \frac{1}{\kappa_n}\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{\log n}} + \frac{1}{\kappa_n}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right),\end{aligned}$$

because $\lim_{n\to\infty}\frac{\sqrt{\log n}}{\kappa_n} = \lim_{n\to\infty}\frac{\sqrt{\log n}\cdot e^{\tau\cdot\sqrt{\log n}}}{n^{1/2}} = \lim_{n\to\infty}\frac{n\cdot e^{\tau\cdot n}}{e^{n^2/2}} = 0$. Due to Step 1, this concludes the proof. □

Proof of Lemma 2. Note that for any $k \ge 1$, if there are at most k undominated Column's actions for some realization $g(m+1,n)$, then there are at most k undominated Column's actions for the corresponding realization $g(m,n)$ constructed from $g(m+1,n)$ by removing the $(m+1)$-th action of Row. Therefore,

$$\Pr(U^C(m+1,n) \le k) \le \Pr(U^C(m,n) \le k) \text{ for any } k = 1,2,\ldots,n.$$

To conclude the proof, $\Pr(U^C(m+1,n) \le 1) = n^{-m} < n^{-(m-1)} = \Pr(U^C(m,n) \le 1)$. □

Proof of Proposition 4. Let's prove three statements in turn.

1. For an arbitrary $m \ge 2$, consider

$$C = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1j} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2j} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{22} & \ldots & c_{mj} & \ldots & c_{mn} \end{pmatrix},$$

where rows $\{c_{1\cdot}, c_{2\cdot}, \ldots, c_{m\cdot}\}$ are *i.i.d.* uniform on S_n. Let $p^C(m,n)$ denote the probability that any particular column is undominated. Note that it is label-independent.

Without loss of generality, focus on the first column $c_{\cdot 1}$. There are n possible values for c_{m1}. Suppose that $c_{m1} = k$ for some $k \in [n]$. It happens with probability $\frac{1}{n}$. Then, the first action is undominated if and only if it is undominated in the $(m-1) \times (n-k+1)$ game formed by removing all columns j with $c_{mj} < k$ and the last row. It happens with probability $p^C(m-1, n-k+1)$. By summing over all possible k, $k \in [n]$, we easily get

$$p^C(m,n) = \frac{\sum_{k=1}^n p^C(m-1, n-k+1)}{n} = \frac{\sum_{k=1}^n p^C(m-1, k)}{n}.$$

To conclude, by the linearity of the expectation, $\mathbb{E}\left[U^C(m,n)\right] = n \cdot p^C(m,n)$.

2. It follows immediately from Lemma 2 and the recurrence relation.

3. *First, let's prove by induction* on $m \geq 2$ that for any $n \geq 1$,

$$\mathbb{E}\left[U^C(m,n)\right] \leq \frac{(\log n)^{m-1}}{(m-1)!} + \frac{(\log n)^{m-2}}{(m-2)!} + \ldots + \frac{(\log n)^2}{2} + \log n + 1.$$

For $m = 2$, $\mathbb{E}\left[U^C(2,n)\right] = H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \leq \log n + 1$ for any $n \geq 1$. Assume that this statement holds for $m \geq 2$. Let's prove it for $m+1$.

For any $x > 0$, define

$$f(x) \equiv \frac{(\log x)^{m-1}}{(m-1)!x} + \frac{(\log x)^{m-2}}{(m-2)!x} + \ldots + \frac{(\log n)^2}{2x} + \frac{\log x}{x} + \frac{1}{x}.$$

This function $f(x)$ is strictly decreasing in $x > 1$. Indeed,

$$\begin{aligned} f'(x) &= \left(-\frac{(\log x)^{m-1}}{(m-1)!x^2} + \frac{(m-1)(\log x)^{m-2}}{(m-1)!x^2}\right) + \ldots + \left(-\frac{\log x}{x^2} + \frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= -\frac{(\log x)^{m-1}}{(m-1)!x^2} < 0 \quad \text{as long as } x > 1. \end{aligned}$$

By the recurrence relation,

$$\mathbb{E}\left[U^C(m+1,n)\right] = \frac{\mathbb{E}\left[U^C(m,1)\right]}{1} + \frac{\mathbb{E}\left[U^C(m,2)\right]}{2} + \ldots + \frac{\mathbb{E}\left[U^C(m,n)\right]}{n}$$
$$\leq f(1)+f(2)+\ldots+f(n) \leq f(1)+\int_1^n f(x)dx = 1+\left(\frac{(\log x)^m}{m!} + \frac{(\log x)^{m-1}}{(m-1)!} + \ldots + \log x\right)\Bigg|_1^n$$
$$= \frac{(\log n)^m}{m!} + \frac{(\log n)^{m-1}}{(m-1)!} + \ldots + \frac{(\log n)^2}{2} + \log n + 1,$$

as desired, where the the first inequality follows from the induction hypothesis and the second one holds because $f(x)$ is strictly decreasing in $x > 1$.

Second, let's prove by induction on $m \geq 2$ that for any $n \geq 1$,

$$\mathbb{E}\left[U^C(m,n)\right] \geq \frac{(\log n)^{m-1}}{(m-1)!}.$$

Note first that for any fixed $m \geq 2$, $p^C(m,n) = \mathbb{E}\left[U^C(m,n)\right]/n$—the probability that the first Column's action is undominated—is decreasing in $n \geq 1$ by its definition.

For $m = 2$, $\mathbb{E}\left[U^C(2,n)\right] = H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n$ for any $n \geq 1$. Assume that the desired statement holds for $m \geq 2$. Let's prove it for $m+1$.

For any $x > 0$, define

$$f(x) \equiv \frac{(\log x)^{m-1}}{(m-1)!x}, \qquad \text{so that} \qquad f'(x) = -\frac{(\log x)^{m-1}}{(m-1)!x^2} + \frac{(m-1)(\log x)^{m-2}}{(m-1)!x^2}$$

is negative for $x > e^{m-1}$ and positive for $x < e^{m-1}$. Therefore, $f(x)$ has a unique (global) maximum at $x_{\max} = e^{m-1}$, so that

$$\max_{x>0} f(x) = f(x_{\max}) = f(e^{m-1}) = \frac{(m-1)^{m-1}}{(m-1)!e^{m-1}} \leq \frac{1}{e} \leq \frac{1}{2},$$

where the first inequality holds because for any $m \geq 2$, by denoting $g(m) \equiv \frac{(m-1)^{m-1}}{(m-1)!e^{m-1}}$, we have $g(m+1) = \left(1 + \frac{1}{m-1}\right)^{m-1} \cdot \frac{g(m)}{e} \leq g(m) \leq g(2) = \frac{1}{e}$.

By our observations above, for any $i \leq \lfloor e^{m-1} \rfloor - 1$, we have

$$\frac{\mathbb{E}\left[U^C(m,i)\right]}{i} = p^C(m,i) \geq p\left(m, \lfloor e^{m-1} \rfloor\right) \geq f\left(\lfloor e^{m-1} \rfloor\right) \geq f(x) \text{ for any } x \leq \lfloor e^{m-1} \rfloor,$$

where the first inequality follows from $p^C(m,n)$ being decreasing in n, the second one holds due to the induction hypothesis, and the third inequality follows because $f(x)$ is increasing for $x < e^{m-1}$. Therefore, for any $i \leq \lfloor e^{m-1} \rfloor - 1$,

$$\frac{\mathbb{E}\left[U^C(m,i)\right]}{i} \geq \int_i^{i+1} f(x)dx.$$

In addition, for $i = 1$, we can even show that

$$\frac{\mathbb{E}\left[U^C(m,1)\right]}{1} - \int_1^2 f(x)dx = 1 - \int_1^2 f(x)dx \geq 1 - \max_{x>0} f(x) \geq \frac{1}{2}.$$

Similarly, for any $i \geq \lceil e^{m-1} \rceil$, we have

$$\frac{\mathbb{E}\left[U^C(m,i)\right]}{i} = p^C(m,i) \geq f\left(i\right) \geq f(x) \text{ for any } x \geq i,$$

where the first inequality follows from the induction hypothesis and the second one holds because $f(x)$ is decreasing for $x > e^{m-1}$. Therefore, for any $i \geq \lceil e^{m-1} \rceil$,

$$\frac{\mathbb{E}\left[U^C(m,i)\right]}{i} \geq \int_i^{i+1} f(x)dx.$$

Finally, for $i = \lfloor e^{m-1} \rfloor$, as long as $m \geq 2$,

$$\int_{\lfloor e^{m-1} \rfloor}^{\lceil e^{m-1} \rceil} f(x)dx \leq \max_{x>0} f(x) \leq \frac{1}{2}.$$

To sum up,

$$\begin{aligned}
\frac{\mathbb{E}\left[U^C(m,1)\right]}{1} &\geq \int_1^2 f(x)dx + \frac{1}{2} \geq \int_1^2 f(x)dx + \int_{\lfloor e^{m-1} \rfloor}^{\lceil e^{m-1} \rceil} f(x)dx, \\
\frac{\mathbb{E}\left[U^C(m,i)\right]}{i} &\geq \int_i^{i+1} f(x)dx \quad \text{for} \quad i \neq \lfloor e^{m-1} \rfloor.
\end{aligned}$$

Therefore, by the recurrence relation and the set of inequalities above,

$$\begin{aligned}\mathbb{E}\left[U^C(m+1,n)\right] &= \frac{\mathbb{E}\left[U^C(m,1)\right]}{1} + \frac{\mathbb{E}\left[U^C(m,2)\right]}{2} + \ldots + \frac{\mathbb{E}\left[U^C(m,n)\right]}{n} \\ &\geq \int_1^{n+1} f(x)dx > \int_1^n f(x)dx = \left(\frac{(\log x)^m}{m!}\right)\Bigg|_1^n = \frac{(\log n)^m}{m!},\end{aligned}$$

as desired. □

Proof of Proposition 6. Let $p(m,n)$ denote the probability that the second Row's action strictly dominates her first one after the first round of Column's elimination. By symmetry and Boole's inequality, we have

$$\Pr\left(S^R(m,n) < m\right) \leq m(m-1)\cdot p(m,n).$$

Without loss of generality, set $c_{1\cdot} = (n, n-1, \ldots, 1)$. Define events E_j, $j \in [n]$, as follows:

$$\begin{aligned}&E_j \equiv \mathcal{C}_j \bigcap \mathcal{R}_j, \quad \text{where} \\ &\mathcal{C}_j \equiv \bigcup_{i\geq 2} \{C : c_{ij} > \max(c_{i,j-1}, c_{i,j-2}, \ldots, c_{i,1})\} \quad \text{and} \quad \mathcal{R}_j \equiv \{R : r_{1j} > r_{2j}\}.\end{aligned}$$

Note that if E_j happens for some $j \in [n]$, then the second Row's action cannot strictly dominate her first one after the first round of Column's elimination. Indeed, if both $\mathcal{C}_j$ and $\mathcal{R}_j$ occur, then the column j stays and $r_{1j} > r_{2j}$.

Assume for the next exercise that events $\{E_j\}_{j\in[n]}$ are mutually independent. Therefore,

$$\begin{aligned}p(m,n) &\leq \Pr\left(\bigcap_{j\in[n]} \overline{E}_j\right) = \prod_{j\in[n]} \left(1 - \Pr\left(E_j\right)\right), \quad \text{where} \\ \Pr\left(E_j\right) &= \frac{1}{2}\cdot\frac{m-1}{j} + \frac{1}{2}\cdot\sum_{k=2}^{m-1}(-1)^{k-1}\binom{m-1}{k}\cdot\left(\frac{1}{j}\right)^k.\end{aligned}$$

By using the inequality $1 - x \leq e^{-x}$ that holds for any $x \geq 0$, we get

$$\prod_{j\in[n]} \left(1 - \Pr\left(E_j\right)\right) \leq e^{-\sum_{j\in[n]}\Pr(E_j)}.$$

Next, we verify that events $\{E_j\}_{j\in[n]}$ are mutually independent. Because rows and columns are mutually independent, it is sufficient to prove that $\{\mathcal{C}_j\}_{j\in[n]}$ are mutually independent.

Note that for any matrix C with $c_{1\cdot} = (n, n-1, \dots, 1)$, we can map it to the matrix $\{m_{ij}\}_{i\in[m-1],\, j\in[n]}$ defined as $m_{ij} = |\{k < j : c_{i+1,k} > c_{i+1,j}\}| \in \{0, 1, \dots, j-1\}$. This mapping is a bijection. Furthermore, $\mathcal{C}_j$ occurs if and only if $m_{ij} = 0$ for some $i \in [m-1]$ (corresponding events are mutually independent).

Finally, we prove two statements in turn.

1. If $m = \mathcal{O}(1)$, then

$$\begin{aligned}\Pr\left(S^R(m,n) < m\right) &\le m(m-1)\cdot p(m,n) \le m(m-1)\cdot e^{-\sum_{j\in[n]}\Pr(E_j)}\\ &= \mathcal{O}(1)\cdot e^{-\frac{m-1}{2}\log n + \mathcal{O}(1)} = \mathcal{O}(1)\cdot n^{-\frac{m-1}{2}} = \mathcal{O}\left(n^{-\frac{m-1}{2}}\right).\end{aligned}$$

2. The statement is trivial for $m = 1$. Consider $m \ge 2$. By applying Bonferroni's inequality up to $k = 2$, we have

$$\Pr(E_j) \ge \frac{m-1}{2}\cdot\frac{1}{j} - \frac{(m-1)(m-2)}{4}\cdot\frac{1}{j^2} = \frac{m-1}{2}\left(\frac{1}{j} - \frac{m-2}{2}\cdot\frac{1}{j^2}\right).$$

As $m \le n$, because $\frac{m-2}{2j} < \frac{1}{2}$ for any $j \ge m-1$,

$$\begin{aligned}\sum_{j\in[n]}\Pr(E_j) &\ge \sum_{j\ge m-1}\Pr(E_j) = \sum_{j\ge m-1}\frac{m-1}{2}\left(\frac{1}{j} - \frac{m-2}{2j}\cdot\frac{1}{j}\right)\\ &> \sum_{j\ge m-1}\frac{m-1}{2}\left(\frac{1}{j} - \frac{1}{2}\cdot\frac{1}{j}\right) = \frac{m-1}{4}\cdot\sum_{j\ge m-1}\frac{1}{j} > \frac{m-1}{4}\cdot\int_m^n\frac{dx}{x} = \frac{m-1}{4}\cdot\log\frac{n}{m},\end{aligned}$$

so that

$$\Pr\left(S^R(m,n) < m\right) \le m(m-1)\cdot e^{-\frac{m-1}{4}\cdot\log\frac{n}{m}} = m(m-1)\cdot\left(\frac{m}{n}\right)^{\frac{m-1}{4}}. \qquad \square$$

Proof of Lemma 3. If either $m = 1$ or $n = 1$, the proof is trivial. Therefore, consider $m, n \ge 2$.

First, let's show that there exists its $(m-1)\times n$ subgame that is strict-dominance solvable. Indeed, if there is a strictly dominated action for Row in the original game, then the $(m-1)\times n$ subgame formed by the exclusion of this action is strict-dominance solvable. Otherwise, there is a strictly dominated action for Column such that in the induced game after the first iteration, Row has a strictly dominated action. Then the $(m-1)\times n$ subgame, formed by the exclusion of this action from the original game, is strict-dominance solvable.

Second, by the symmetric argument, there exists its $m \times (n-1)$ subgame that is strict-dominance solvable.

Finally, we can repeat the previous steps to prove the desired result by induction. □

Proof of Proposition 7. The idea of this proof is to estimate the probability to eliminate at least $\frac{n}{3}$ rows (actions of Row) or columns (actions of Column). This probability will provide the desired upper bound for the probability of strict-dominance solvability.

Start the standard iterative elimination procedure and stop exactly when at least $\frac{n}{3}$ rows or at least $\frac{n}{3}$ columns are deleted. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial. Without loss of generality, suppose that we deleted $\frac{n}{3}$ rows and at most $\frac{n}{3}$ columns.

Let X be the set of columns that are not yet eliminated. Similarly, Y is defined as the set of rows that are not yet deleted. Their complements $X' \equiv [n]\backslash X$ and $Y' \equiv [n]\backslash Y$ correspond to eliminated columns and rows, respectively. By the previous paragraph, $|X| \geq \frac{2n}{3}$ and $|Y| = \frac{2n}{3}$. Also, for any row r_i eliminated, $i \in Y'$, there must exist a row r_j not eliminated yet, $j \in Y$, so that $r_{jx} > r_{ix}$ for any $x \in X$, namely r_j strictly dominates r_i when restricted to columns X.

For any row r_i eliminated, $i \in Y'$, choose some row $r_{j(i)}$ not eliminated yet, $j(i) \in Y$, so that $r_{j(i)x} > r_{ix}$ for any $x \in X$, and draw a directed edge from $j(i)$ to i. We get a collection of r stars of sizes $k_1, k_2, \ldots, k_r$ with centers in Y (not eliminated rows) and leaves Y' (eliminated rows), so that $k_1 + k_2 + \ldots + k_r = |Y'| = \frac{n}{3}$.

First, the total number of ways to choose such X, Y, and stars, is bounded above by

$$\binom{n}{|X|} \cdot \binom{n}{|Y|} \cdot |Y|^{|Y'|} \leq 2^n \cdot 2^n \cdot |Y|^{n-|Y|} \leq 4^n \cdot \left(\frac{2n}{3}\right)^{\frac{n}{3}}.$$

Second, for any such fixed X, Y, and r stars of sizes $k_1 + k_2 + \ldots + k_r = |Y'| = \frac{n}{3}$, the probability that for each star, its center dominates all corresponding leaves when restricted to X, is exactly

$$\left(\frac{1}{k_1+1} \cdot \frac{1}{k_2+1} \cdot \ldots \cdot \frac{1}{k_r+1}\right)^{|X|} \leq \left(\frac{1}{|Y'|+1}\right)^{|X|} \leq \left(\frac{1}{|Y'|}\right)^{|X|} \leq \left(\frac{1}{n/3}\right)^{\frac{2n}{3}}.$$

Based on two previous inequalities, the probability to eliminate at least $\frac{n}{3}$ rows or columns is bounded above by

$$4^n \cdot \left(\frac{2n}{3}\right)^{\frac{n}{3}} \cdot \left(\frac{1}{n/3}\right)^{\frac{2n}{3}} = n^{-\left(\frac{1}{3}-o(1)\right)n}. \qquad \square$$

Proof of Proposition 8. There are two relevant cases to consider. First, if $m \geq n^{0.9}$, then by Lemma 3 and Proposition 7,

$$\pi(m,n) \leq \binom{n}{m} \cdot \pi(m,m) \leq \binom{n}{m} \cdot \frac{1}{m^{0.3m}} \leq \binom{n}{m} \cdot \frac{1}{n^{0.27m}}.$$

By using the standard upper bound for the binomial coefficient, we get

$$\binom{n}{m} \cdot \frac{1}{n^{0.27m}} \leq \left(\frac{en}{m}\right)^m \cdot \frac{1}{n^{0.27m}} = e^m \cdot \frac{1}{n^{0.17m}}.$$

Second, if $m \leq n^{0.9}$, then by Proposition 6,

$$\pi(m,n) \leq m^2 \cdot \left(\frac{1}{n^{0.1}}\right)^{\frac{m-1}{4}} \leq m^2 \cdot \frac{1}{n^{0.025m}}.$$

To sum up, by taking $C_2 = 0.01$ and sufficiently large $C_1 > 0$, for any $m \leq n$, we have $\pi(m,n) \leq C_1 n^{-C_2 m}$, as desired. $\square$

APPENDIX – ENUMERATIVE ISSUES

Consider the case of $m = 3$ and the basic problem of finding the probability that Column has no strictly dominated actions. If we fix $c_{1\cdot} = e_n$, there are no strictly dominated actions for Column if and only if a pair of permutations $(c_{2\cdot}, c_{3\cdot})$ avoids the permutation pattern in which $c_{2j} > c_{2i}$ and $c_{3j} > c_{3i}$ for some $j > i$. This particular avoidance imposes restrictions on the pair of *i.i.d.* uniform $(c_{2\cdot}, c_{3\cdot})$ that has been the main object of interest in Hammett and Pittel (2008). Formally, our problem can be equivalently reformulated in terms of what is termed *parallel permutation patterns* in enumerative combinatorics (see our Lemma 8 below). This problem lies at the research frontier of that literature (Hammett and Pittel, 2008; Gunby and Pálvölgyi, 2019). To make things worse, in general, permutation patterns induced by strict dominance are different from those studied in the literature on permutation avoidance.

As a numerical demonstration, Table 1 displays the number of possible Column's matrices with one fixed payoff row for $m = 3$ and $n \in [6]$ corresponding to exactly k strictly undominated actions, $k \in [n]$.[21] These numbers can be viewed as a generalization of the unsigned Stirling numbers of the first kind for $m = 3$. In particular, the underlined sequence corresponds to the number of incidents in which Column has no strictly dominated actions, as described above. The table suggests properties similar to those of the standard Stirling numbers. Values appear to be log-concave (and unimodal) and asymptotically normal with faster convergence rates. This already hints at the qualitative similarities between the general m by n case and the particular 2 by n case studied above.

n \ k	1	2	3	4	5	6
1	$\underline{1}$					
2	1	$\underline{3}$				
3	4	15	$\underline{17}$			
4	36	147	242	$\underline{151}$		
5	576	2460	4775	4690	$\underline{1899}$	
6	14400	63228	134909	164193	109959	$\underline{31711}$

Table 1: $(n!)^2 \cdot \Pr(U^C(3, n) = k)$, $k \in [n]$: exact calculations (the underlined sequence corresponds to sequence A007767 in the OEIS)

The combinatorics community has accumulated knowledge of many number sequences summarized in the On-Line Encyclopedia of Integer Sequences (OEIS). It is worth noting that none of the sequences corresponding to any dimension of our analysis has been enumerated before in the OEIS. This suggests that our fundamental combinatorial object has not been studied previously.

To state Lemma 8, we need to introduce three additional definitions related to the literature on permutation patterns. First, we say that for $\sigma_1, \dots, \sigma_d \in S_n$ and $\sigma'_1, \dots, \sigma'_d \in S_m$, $(\sigma_1, \dots, \sigma_d)$ *avoids* $(\sigma'_1, \dots, \sigma'_d)$ if there does not exist indices $c_1 < \cdots < c_m$ such that $\sigma_i(c_1)\,\sigma_i(c_2) \cdots \sigma_i(c_m)$ is order-isomorphic to σ'_i for all i (e.g., see Gunby and Pálvölgyi, 2019; Klazar, 2000). Second, we say that $\pi \preceq \sigma$ in the *weak Bruhat Order* if there is a chain $\sigma = \omega_1 \to \omega_2 \to \cdots \to \omega_s = \pi$, where each ω_t is a *simple reduction* of ω_{t-1}, i.e. obtained from ω_{t-1} by transposing two *adjacent* elements $\omega_{t-1}(i), \omega_{t-1}(i+1)$ with $\omega_{t-1}(i) > \omega_{t-1}(i+1)$. Equivalently (see Lemma 4.1 in Hammett and Pittel, 2008), $\pi \preceq \sigma$ in the *weak Bruhat Order* if $I(\pi) \subseteq I(\sigma)$, where for any $\omega \in S_n$, the inversion set $I(\omega) = \{(i, j) \mid i < j \text{ with } \omega^{-1}(i) > \omega^{-1}(j)\}$ is defined to be the set of all inversions in ω.

[21]These numbers correspond to exact computations for all $(n!)^2$ possible combinations.

Finally, for any $\sigma \in S_n$, let $\sigma^\star \in S_n$ denote its *complement*, i.e. $\sigma^\star(i) = n + 1 - \sigma(i)$.

Lemma 8. *Consider a random game $G(3, n)$. Then, for any $n \geq 1$,*

$$\prod_{i=1}^{n}(H_i/i) \leq \Pr(U^C(3,n) = n) = \Pr\left((c^\star_{3\cdot})^{-1} \preceq c^{-1}_{2\cdot}\right) \leq (0.362)^n.$$

Proof. As in Lemma 1, we can set $c_{1\cdot}$ to $e_n \equiv (1, 2, \ldots, n)$. This is without loss of generality.

Note first that all Column's actions are undominated if and only if $(c_{2\cdot}, c_{3\cdot})$ avoids $(12, 12)$. It holds whenever for any $i < j$ with $c_{2i} < c_{2j}$, then we must have $c_{3i} > c_{3j}$, or equivalently $c^\star_{3i} < c^\star_{3j}$. In other words, the set $I(c^{-1}_{2\cdot})$ of inversions of $c_{2\cdot}$ contains the set $I\left((c^\star_{3\cdot})^{-1}\right)$ of inversions of $(c^\star_{3\cdot})^{-1}$, i.e. $I\left((c^\star_{3\cdot})^{-1}\right) \subseteq I(c^{-1}_{2\cdot})$. It happens if and only if $(c^\star_{3\cdot})^{-1} \preceq c^{-1}_{2\cdot}$.

Note that trivially $\Pr\left((c^\star_{3\cdot})^{-1} \preceq c^{-1}_{2\cdot}\right) = \Pr\left(\pi \preceq \sigma\right)$, where $\sigma, \pi \in S_n$ are selected independently and uniformly at random. Probability bounds for this problem have been studied by Hammett and Pittel (2008). $\square$

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