Supplementary Material for "Task Allocation and On-the-job Training"

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Abstract

We illustrate the comparative statics pertaining to the perfect-monitoring setting studied in the paper.

1 Comparative Statics with Perfect Monitoring

Proposition 3 in the main text characterizes the equilibrium policy (q_P^e, μ_P^e) and the optimal policy (q_P^*, μ_P^*) in the discretionary and centralized settings, respectively, when monitoring is perfect. We now consider the impact of changes in θ, λ, and the training technology on these outcomes. By and large, comparative statics are similar to those observed with limited monitoring. For exposition simplicity, we restrict attention to a linear training technology, $f(x) = ax$ for some $a > 0$.

As θ grows, either through an increase in the relative benefit $h - l$ of service by seniors, or through a decrease in waiting costs c, queueing for senior service becomes relatively more attractive. Clients then seek more senior service when choosing on their own or when directed by a planner. As in the limited-monitoring case, this translates into higher average quality and lower training. It also increases the average wait times in the senior queue.[1]

As clients' arrival rate λ or training efficacy a increase, the feasibility constraint is affected, making the analysis more subtle. While the indifference condition in (9) of the main text does not change, the first-order condition in (10) does change with λ. As we show, increases in λ and a yield increased equilibrium thresholds and more training.

[1]For our discretionary setting, these conclusions hold for general production technologies.

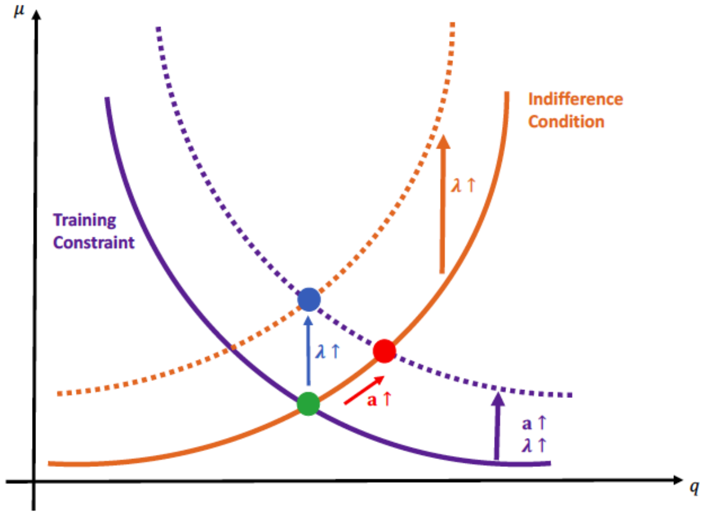

Figure 1: Comparative Statics for Discretionary Settings with Perfect Monitoring for Training Technology $f(x) = ag(x)$

Proposition A1 (Perfect Monitoring – Comparative Statics) *The following comparative statics hold:*

1. *As $h - l$ increases, or c decreases, q_P^e and q_P^* increase, while the induced masses of seniors, μ_P^e and μ_P^*, decrease.*
2. *As λ increases, k_P^e, μ_P^e, q_P^*, and μ_P^* increase.*
3. *As a increases, k_P^e, q_p^e, μ_P^e, and μ_P^* increase.*

While changes in $h-l$ or c have similar impacts as those observed in the limited-monitoring case, we see different patterns when it comes to arrival rates and technology efficacy. Consider first the discretionary setting with training technology of the form $f(x) = ag(x)$, with $a > 0$. When λ or a increase, in the space of (q, μ), the graph corresponding to the indifference condition for each λ, $G_1 = \{(q, \mu) : k(q, \mu; \lambda) = \mu\theta + 1\}$ shifts up as λ increases and does not change with changes in a, see Figure 1. The graph $G_2 = \{(q, \mu) : \mu = ag((1 - q)\lambda))\}$, corresponding to the training constraint, shifts up with increases in both λ and a, also depicted in Figure 1. Consequently, with increases in a, we have that k_P^e, q_P^e, and μ_P^e increase. However, with increases in λ, while k_P^e and μ_P^e increase, q_P^e may go up or down. Which way q_P^e shifts depends on the details of the training technology, which affects how G_2 moves relative to G_1 as λ increases, and thereby determines their intersection.

For the centralized allocation, as θ increases, the training constraint remains constant, while the first, linear term of the planner's objective increases. It follows from the proof of Proposition 3 that the optimal q_P^* increases, while μ_P^* decreases.

Understanding the impact of λ and a on the planner's solution requires somewhat different techniques. Consider the graphs of the training constraint and the planner's first-order condition, which links $\phi \equiv \lambda/\mu$ and q. As Proposition 3 suggests, at the optimal allocation policy, the training constraint and the planner's objective are tangent to one another. The arrival rate λ has no (explicit) impact on the training constraint that, when $f(x) = ax$, can simply be written as $\phi = 1/a(1-q)$. Increases in λ, however, alter the planner's objective so as to generate the result. Intuitively, small increases in q raise the planner's objective more as λ increases, yielding the increase in q_P^*. The training constraint then suggests that the level of ϕ at the optimal allocation increases with q as well. In fact, since the training constraint, put in terms of ϕ and q, is convex, small changes in q have more than a linear impact on the resulting levels of $\phi = \lambda/\mu$. In fact, we show that the increase in ϕ is greater than the increase in λ that generated it. It follows that μ_P^* increases with λ. A similar intuition holds when considering changes in a. An increase in a impacts the training constraint, attenuating the marginal effects of increases in q on ϕ. Such changes do not have a direct effect on the planner's objective. This can be shown to decrease the value of ϕ at the point of tangency, leading to an increase in μ_P^*.[2]

Proof of Proposition A1: First, consider the discretionary setting. The equilibrium (q_P^e, μ_P^e) is identified as the intersection of two graphs:

$$G_1 = \{(q,\mu) : \mu = a \cdot g((1-q)\lambda)\} \quad \text{and} \quad G_2 = \{(q,\mu) : k(q,\mu;\lambda) = \mu\theta + 1\}.$$

In the proof of Proposition 3, we have shown that G_1 is downward sloped and G_2 is upward sloped. The graph G_1 shifts right if either λ or a increase and is unchanged if θ increases. The threshold $k(q,\mu,\lambda)$ is strictly decreasing in μ and strictly increasing in q and λ. Hence, the graph G_2 shifts to the left with an increase in λ, shifts to the right with an increase in θ, and remains unchanged with an increase in a.

The impacts of changes in θ, and therefore those of changes in $h-l$ or c, follow immediately. If λ increases, the equilibrium μ_P^e increases, which implies that k_P^e increases by the indifference condition, but q_P^e may go up or down. Last, if a increases, q_P^e and μ_P^e increase, and by the

[2] The challenge in signing q_P^* arises since, intuitively, increases in a and the resulting μ_P^* impact q_P^* in different ways. Namely, the training constraint requires that $q = 1 - \phi/a$ and signing the impact of increases in a on ϕ^*/a are difficult to identify.

indifference condition, k_P^e increases.

Second, consider the centralized setting with a linear training technology $f(x) = ax$,with $a > 0$. It is useful to consider the space of (q, ϕ), where $\phi \equiv \frac{\lambda}{\mu} = \frac{1}{a(1-q)}$ does not depend on λ. From the proof of Proposition 3, recall that $\mathbf{E}[Q]$ in $[P']$ is continuously differentiable at $\phi = 1$, and ϕ is restricted to be in $[\underline{\phi}, 1+1/a)$. It is straightforward to show that the first-order condition of an interior optimal solution is $\frac{d\mathbf{E}[Q]}{d\phi} = \frac{\lambda\theta}{a\phi^2}$.

If either λ or θ increase, ϕ_P^* has to increase, since $\mathbf{E}[Q]$ is a convex function of ϕ, which implies that the derivative $\frac{d\mathbf{E}[Q]}{d\phi}$ increases in ϕ. If θ increases, $\mu_P^* = \lambda/\phi_P^*$ decreases. On the other hand, suppose that λ increases. We can rewrite the first-order condition above as $\left(\frac{d\mathbf{E}[Q]}{d\phi}\right)\phi = \frac{\lambda\theta}{a\phi} = \frac{\mu\theta}{a}$. Since ϕ_P^* increases, the left-hand side of the equality increases, which implies that μ_P^* increases, and q_P^* increases as well because of the training constraint $\phi_P^* = \frac{1}{a(1-q_P^*)}$.

Last, we show that ϕ_P^* decreases in a, implying that μ_P^* increases in a. Consider any a such that $\phi_P^* \neq 1$ is an interior solution. The optimal ϕ_P^* satisfies the first-order condition:

$$\frac{\lambda\theta}{a\phi^2} - \frac{\phi(2-\phi)}{(1-\phi)^2} - \frac{\log(a(1-\phi)+1))(-\log\phi + (1/\phi) - 1)}{a(1-\phi)^2(\log\phi)^2} = 0 \tag{1}$$
$$\iff w(\phi; a) \equiv \frac{\lambda\theta}{\phi^2} - \frac{a\phi(2-\phi)}{(1-\phi)^2} - \frac{\log(a(1-\phi)+1))(-\log\phi + (1/\phi) - 1)}{(1-\phi)^2(\log\phi)^2} = 0.$$

By the Implicit Function Theorem $\frac{d\phi}{da} = -\frac{dw/da}{dw/d\phi}$. Also, we showed in the proof of Proposition 3 that the objective function of $[P']$ is strictly concave in ϕ. That is, $\frac{dw(\phi,a)}{d\phi} < 0$ at every $\phi \in (\underline{\phi}, 1) \cup (1, 1+1/a)$. Hence, the following claim is sufficient to conclude the proof of Proposition A1.

Claim A1: *For any* (ϕ, a) *such that* $\phi \in (\underline{\phi}, 1) \cup (1, 1+1/a)$, $\frac{dw(\phi,a)}{da} < 0$.

Proof of Claim A1: Observe that

$$\phi \geq \underline{\phi} = \frac{1+\sqrt{1+4a}}{2a} \iff (2a\phi - 1)^2 \geq 1 + 4a \iff a \geq \frac{1+\phi}{\phi^2}.$$

From (1), we get

$$\begin{aligned}
\frac{dw(\phi,a)}{da} &= -\frac{\phi(2-\phi)}{(1-\phi)^2} + \frac{1}{(1-\phi)(\log\phi)(a(1-\phi)+1)} \\
&\quad - \frac{a}{(\log\phi)(a(1-\phi)+1)^2} + \frac{-(\log\phi) + (1/\phi) - 1}{(1-\phi)^2(\log\phi)^2}\frac{1-\phi}{a(1-\phi)+1} \\
&= -\frac{\phi(2-\phi)}{(1-\phi)^2} - \frac{a}{(\log\phi)(a(1-\phi)+1)^2} + \frac{1}{\phi(\log\phi)^2(a(1-\phi)+1)}.
\end{aligned}$$

To show that $\frac{dw(\phi,a)}{da} < 0$, we distinguish between three cases. First, if $\phi \geq 2$, we multiply $\frac{dw(\phi,a)}{da}$ by $-(1-\phi)(\log\phi)(a(1-\phi)+1) > 0$, and obtain

$$\begin{aligned}
&-(1-\phi)(\log\phi)(a(1-\phi)+1)\frac{dw}{da} \\
&= a\phi(2-\phi)(\log\phi) + \frac{\phi(2-\phi)(\log\phi)}{1-\phi} + \frac{a(1-\phi)}{a(1-\phi)+1} - \frac{1-\phi}{\phi(\log\phi)},
\end{aligned}$$

which is strictly decreasing in a. Hence, we obtain an upper bound of the above expression by substituting a with its lower bound $\frac{1+\phi}{\phi^2}$. The upper bound, which is a function of ϕ only, is less than -2.278 for every $\phi \geq 2$.

If $1 < \phi < 2$, we have

$$\begin{aligned}
(a(1-\phi)+1)^2\frac{dw}{da} &= -\frac{\phi(2-\phi)(a(1-\phi)+1)^2}{(1-\phi)^2} - \frac{a}{\log\phi} + \frac{a(1-\phi)+1}{\phi(\log\phi)^2} \\
&= \left(-a^2\phi(2-\phi) - \frac{2a\phi(2-\phi)}{1-\phi} - \frac{\phi(2-\phi)}{(1-\phi)^2}\right) - \frac{a}{\log\phi} + \left(\frac{a(1-\phi)}{\phi(\log\phi)^2} + \frac{1}{\phi(\log\phi)^2}\right) \\
&= -a^2\phi(2-\phi) - a\left(\frac{2\phi(1-\phi)}{1-\phi} + \frac{1}{\log\phi} - \frac{1-\phi}{\phi(\log\phi)^2}\right) - \frac{\phi(2-\phi)}{(1-\phi)^2} + \frac{1}{\phi(\log\phi)^2}.
\end{aligned}$$

For any $1 < \phi < 2$, $\frac{2\phi(1-\phi)}{1-\phi} + \frac{1}{\log\phi} - \frac{1-\phi}{\phi(\log\phi)^2} > 2.435$, so the above expression is strictly decreasing in a. Substituting a with its lower bound $\frac{1+\phi}{\phi^2}$ results in an upper bound that is a function of ϕ only, and is lower than -0.82.

Finally, if $\phi < 1$, we have

$$\begin{aligned}
\frac{(a(1-\phi)+1)^2}{a}\frac{dw}{da} &= -\frac{\phi(2-\phi)}{(1-\phi)^2}\left(a(1-\phi)^2 + 2(1-\phi) + \frac{1}{a}\right) - \frac{1}{\log\phi} + \frac{1}{\phi(\log\phi)^2}\left(1-\phi+\frac{1}{a}\right) \\
&= -a\phi(2-\phi) + \frac{1}{a}\left(\frac{1}{\phi(\log\phi)^2} - \frac{\phi(2-\phi)}{(1-\phi)^2}\right) - \frac{2\phi(2-\phi)}{1-\phi} - \frac{1}{\log\phi} + \frac{1-\phi}{\phi(\log\phi)^2}.
\end{aligned}$$

For any $\phi < 1$, $\frac{1}{\phi(\log\phi)^2} - \frac{\phi(2-\phi)}{(1-\phi)^2} > \frac{13}{12}$, so the above expression is strictly decreasing in a. Substituting a with its lower bound $\frac{1+\phi}{\phi^2}$ results in an upper bound that is a function of ϕ, and less than $-\frac{47}{24}$. This concludes the proof of Claim A1 and Proposition A1. ∎