Supplementary Material for "Collective Progress: Dynamics of Exit Waves"

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Abstract

This appendix contains: *i*) a discussion of continuity of the stopping boundaries; *ii*) proofs of corollaries in the main text; *iii*) a recursive formulation of the welfare function; *iv*) the equilibrium and social planner's policies for exponential well-ordered cost functions; *v*) equilibrium characterization pertaining to the case when later innovations are penalized; *vi*) welfare comparisons for fixed search scopes; *vii*) equilibrium characterization pertaining to general search technologies; and *viii*) analysis of the model with independent observations.

1 Continuity of the Stopping Boundaries

In this section we show that, in any Markov equilibrium, any alliance is associated with a boundary that is almost surely continuous, provided minimal conditions on agents' individual searches hold. Namely, we assume that all agents search non-trivially if on their own, effectively assuming that all search costs are not prohibitively high.

Lemma OA 1. *If for all $i \in N$, $g_i^i(M) < M$ for all M, then in any Markov equilibrium, for any $A \subset N$, there exists an optimal stopping boundary $g^A : \mathbb{R}_+ \to \mathbb{R}$ such that $g^A(M) \leq M$, g^A is almost surely continuous, and $\tau^A = \inf\{t \geq 0 : X_t = g^A(M_t)\}$.*

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Proof. First, for any maximal value M, never stopping ($g^A(M) = -\infty$) cannot be an equilibrium strategy for any alliance. Otherwise, the payoffs to all members would be unboundedly small, violating individual optimality. Furthermore, since the single-agent search value is positive for all agents, it can never be a best response, under any equilibrium and for any agent, to stop when $X = M$. Therefore, the stopping boundary of any alliance lies below M.

Towards a contradiction, suppose that, for some alliance A, the stopping boundary $g^A(M)$ is discontinuous. Let $\hat{M}$ denote a point of discontinuity. Since $g^A(\hat{M})$ is a point of stopping, we must have some agent i that finds it optimal to stop at $g^A(\hat{M})$. That is, $g_i^A(\hat{M}) = g^A(\hat{M})$. By lemma $A4$ in the main text, the stopping boundary of agent i has to be a drawdown stopping boundary, and therefore continuous. Hence, in order to have a discontinuity in g^A, it must be that the identity of the agent who stops changes from i some $j \neq i$ around $\hat{M}$. Without loss of generality, assume that the change happens to the right of $\hat{M}$. That is, for $\varepsilon > 0$, at $\hat{M}+\varepsilon$ agent j is the first to leave alliance A. Again, by lemma $A4$, agent j's stopping boundary is also a drawdown stopping boundary, and thus continuous. Furthermore, since both boundaries are drawdown boundaries, they are parallel and never cross one another. Therefore, the agent with the lower drawdown would prefer to stop at both $\hat{M}$ and $\hat{M} + \varepsilon$ in a continuous manner, contradicting the existence of a discontinuity at $\hat{M}$. In other words, in every alliance, there is at least one agent that has a continuous boundary, and the stopping boundary of the alliance is that boundary. Since other drawdowns are never reached, it is possible to have discontinuities in other agents' stopping boundaries. However, since those are never reached, it is without loss to assume they also use the calculated drawdowns. ■

2 Proofs of Corollaries

Proof of Corollary 1. First, we show as alliances shrink, total search scope cannot increase. Suppose not and assume there exists an alliance A and an agent i such that $A \cup \{i\}$ generates lower overall search scope compared to that generated by alliance A. That is, $\sigma^A > \sigma^{A\cup\{i\}}$. From uniqueness of interior solutions, for each agent $j \in A$, we must have

$$\frac{2c_j(\sigma_j^A)}{c_j'(\sigma_j^A)} = \sigma^A \quad \text{and} \quad \frac{2c_j(\sigma_j^{A\cup\{i\}})}{c_j'(\sigma_j^{A\cup\{i\}})} = \sigma^{A\cup\{i\}}.$$

Since $\sigma^A > \sigma^{A\cup\{i\}}$ and each of the cost functions is log-convex, it must be the case that $\sigma_j^A \leq \sigma_j^{A\cup\{i\}}$. Thus,

$$\sum_{j\in A}\sigma_j^A \leq \sum_{j\in A}\sigma_j^{A\cup\{i\}} < \sum_{j\in A}\sigma_j^{A\cup\{i\}} + \underline{\sigma} \leq \sigma^{A\cup\{i\}},$$

in contradiction.

The comparative statics pertaining to individuals' search scope follows immediately from log-convexity of the cost functions. ■

The proof of Corollary 2 follows directly from the text preceding it.

Proof of Corollary 3. To prove the corollary, we introduce superscripts *eq* and *sp* to denote the equilibrium and social planner's solution, respectively (these are suppressed otherwise, when there is low risk of confusion). We use the following set of lemmas. For these, we assume interior equilibrium and social planner search scopes, as presumed in the corollary.

Lemma OA 2. *Any active alliance A has a higher search scope under the social planner's solution compared to the equilibrium.*

Proof. Towards a contradiction, suppose there exists an alliance A such that $\sum_{i\in A}\sigma_i^{A,eq} > \sum_{i\in A}\sigma_i^{A,sp}$. The social planner's solution satisfies

$$\frac{2\sum_{i\in A} c_i(\sigma_i^{A,sp})}{c_j'(\sigma_j^{A,sp})} = \sum_{i\in A}\sigma_i^{A,sp} \quad \forall j \in A,$$

which implies that

$$\frac{2c_j(\sigma_j^{A,sp})}{c_j'(\sigma_j^{A,sp})} < \sum_{i\in A}\sigma_i^{A,sp} < \sum_{i\in A}\sigma_i^{A,eq} \quad \forall j \in A.$$

From log-convexity of costs, the left-hand side of this inequality increases as $\sigma_j^{A,sp}$ decreases. For the equilibrium constraint to hold, $\sigma_j^{A,eq} < \sigma_j^{A,sp}$ for all $j \in A$, in contradiction. ■

Lemma OA 3. *In the social planner's solution, for any alliance A and any agent $i \notin A$, $\sigma^{A,sp} < \sigma^{A\cup\{i\},sp}$.*

Proof. Suppose not. Then, there exists an alliance A and an individual $i \notin A$ such that $\sigma^{A,sp} \geq \sigma^{A\cup\{i\},sp}$. Then, for all $l \in A$, we must have

$$\frac{2\sum_{j\in A} c_j(\sigma_j^{A,sp})}{c_l'(\sigma_l^{A,sp})} = \sigma^{A,sp} \quad \text{and} \quad \frac{2\sum_{j\in A\cup\{i\}} c_j(\sigma_j^{A\cup\{i\},sp})}{c_l'(\sigma_l^{A_k\cup\{i\},sp})} = \sigma^{A\cup\{i\},sp}.$$

Since search costs are strictly positive, the second equality implies that, for all $l \in A$,

$$\frac{2\sum_{j\in A} c_j(\sigma_j^{A\cup\{i\},sp})}{c_l'(\sigma_l^{A\cup\{i\},sp})} < \frac{2\sum_{j\in A\cup\{i\}} c_j(\sigma_j^{A\cup\{i\},sp})}{c_l'(\sigma_l^{A\cup\{i\},sp})} = \sigma^{A\cup\{i\},sp} \leq \sigma^{A,sp}.$$

Log-convexity of costs implies that the left-hand side of this inequality increases as $\sigma_l^{A\cup\{i\},sp}$ decreases. The social planner's constraint for alliance A then implies that $\sigma_l^{A,sp} < \sigma_l^{A\cup\{i\},sp}$ for all $l \in A$, in contradiction to the last inequality. ■

Lemma OA 4. *In the social planner's solution if A_k and A_{k+1} are consecutive active alliances in the social planner's solution, then for any i in A_{k+1}, we have $\sigma_i^{A_k,sp} > \sigma_i^{A_{k+1},sp}$.*

Proof. From Proposition 4, if A_k and A_{k+1} are part of the optimal sequence, we have

$$\frac{\sum_{j\in A_k} c_j(\sigma_j^{A_k,sp})}{(\sigma^{A_k,sp})^2} - \frac{\sum_{j\in A_{k+1}} c_j(\sigma_j^{A_{k+1},sp})}{(\sigma^{A_{k+1},sp})^2} > 0.$$

Furthermore, from Proposition 3, for all $i \in A_{k+1}$,

$$\frac{2\sum_{j\in A_k} c_j(\sigma_j^{A_k,sp})}{c_i'(\sigma_i^{A_k,sp})} = \sigma^{A_k,sp} \quad \text{and} \quad \frac{2\sum_{j\in A_{k+1}} c_j(\sigma_j^{A_{k+1},sp})}{c_i'(\sigma_i^{A_{k+1},sp})} = \sigma^{A_{k+1},sp}.$$

Combining these yields, for all $i \in A_{k+1}$,

$$\frac{(c_i'(\sigma_i^{A_k,sp}))^2}{\sum_{j\in A_k} c_j(\sigma_j^{A_k,sp})} - \frac{(c_i'(\sigma_i^{A_{k+1},sp}))^2}{\sum_{j\in A_{k+1}} c_j(\sigma_j^{A_{k+1},sp})} = \frac{c_i'(\sigma_i^{A_k,sp})}{2\sigma^{A_k,sp}} - \frac{c_i'(\sigma_i^{A_{k+1},sp})}{2\sigma^{A_{k+1},sp}} > 0.$$

Since $\sigma^{A_k,sp} > \sigma^{A_{k+1},sp}$ from the previous lemma, for all $i \in A_{k+1}$, we must have $c_i'(\sigma_i^{A_k,sp}) > c_i'(\sigma_i^{A_{k+1},sp})$, which in turn implies $\sigma_i^{A_k,sp} > \sigma_i^{A_{k+1},sp}$. ■

To prove Corollary 3, we combine the three lemmas with Corollary 1. Consider any non-singleton alliance A on path for the equilibrium and the social planner's solution. For any $i \in A$, Corollary 1 implies that $\sigma_i^{A,eq} < \sigma_i^{\{i\},eq}$. From the lemmas above, $\sigma_i^{A,sp} > \sigma_i^{\{i\},sp}$. Since an individual searching on her own chooses the optimal search scope, $\sigma_i^{\{i\},eq} = \sigma_i^{\{i\},sp}$. We therefore have $\sigma_i^{A,sp} > \sigma_i^{A,eq}$. Furthermore, from Lemma OA 4, in the welfare maximizing solution, each agent's search scope decreases as her alliance shrinks in size. ■

Proof of Corollary 4. Towards a contradiction, suppose that for some alliance A_k, which is active on both the social planner's solution and the equilibrium, we have $d_{A_k}^{eq} > d_{A_k}^{sp}$. Consider an alternative policy for the social planner under which each agent i in $A_k \setminus A_{k+1}$ searches with $\sigma_i^{A_k,eq}$ when the current gap $M - X$ is between d^{eq} and $d_{A_k}^{sp}$. Under this policy, each agent i in A_{k+1} still searches using a scope $\sigma_i^{A_{k+1},sp}$, as in the candidate policy. We now show this generates an improvement.

First, under this policy, agents in $A_k \setminus A_{k+1}$ are better off. Indeed, those agents are utilizing the same search scope they would in equilibrium. Agents in A_{k+1} are searching with scope $\sigma_i^{A_{k+1},sp}$. From Corollary 3, $\sigma_i^{A_{k+1},sp} > \sigma_i^{A_k,eq}$. Thus, agents in $A_k \setminus A_{k+1}$ are receiving greater positive externalities compared to equilibrium. Furthermore, since the gap is larger than $d_{A_k}^{eq}$, in equilibrium the agents have a positive continuation value, which is increased due to positive externalities.

Second, under this policy, agents in A_{k+1} are better off. Indeed, when the gap between the observed maximum and their search outcomes falls between $d_{A_k}^{sp}$ and $d_{A_k}^{eq}$, their own search scope is unchanged, but they receive positive externalities from agents in $A_k \setminus A_{k+1}$.

We conclude that $d_{A_k}^{eq} > d_{A_k}^{sp}$ cannot be optimal for any A_k.

■

3 Recursive Formulation of Welfare

As mentioned in the text, welfare can be written in a recursive fashion. Namely, we have:

Proposition OA 1. *Suppose $A_1, \ldots, A_K$ is the optimal sequence of alliances with associated drawdown sizes $d_{A_1}, ..., d_{A_K}$. When search starts at $X_0 = M_0 = 0$, we have:*

$$W(0,0,N) = \sum_{m=1}^{K} \left((d_{A_m})^2 - (d_{A_{m-1}})^2\right) \frac{\sum_{i \in A_m} c_i(\sigma_i^{A_m})}{(\sigma^{A_m})^2},$$

where we set $d_{A_0} = 0$.

Proof. Using Propositions 3 and 4, we can write the expected welfare as follows:

$$\begin{aligned}
W_k(0,0) =& (d_{A_k})^2 \frac{\sum_{i \in A_k} c_i(\sigma_i^{A_k})}{(\sigma^{A_k})^2} + d_{A_k} \sum_{m=k}^{K} (d_{A_{m+1}} - d_{A_m}) \frac{2\sum_{i \in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_m})^2} \\
& + (d_{A_{k+1}} - d_{A_k})^2 \frac{\sum_{i \in A_{k+1}} c_i(\sigma_i^{A_{k+1}})}{(\sigma^{A_{k+1}})^2} \\
& + (d_{A_{k+1}} - d_{A_k}) \sum_{m=k+1}^{K} (d_{A_{m+1}} - d_{A_m}) \frac{2\sum_{i \in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_m})^2} \\
& \vdots
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&+(d_{A_K} - d_{A_{K-1}})^2 \frac{\sum_{i \in A_{K-1}} c_i(\sigma_i^{A_{K-1}})}{(\sigma^{A_{K-1}})^2} \\
&\quad + (d_{A_K} - d_{A_{K-1}}) \sum_{m=K-1}^{K} (d_{A_{m+1}} - d_{A_m}) \frac{2 \sum_{i \in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_m})^2} \\
&+(d_{A_K} - d_{A_{K-1}})^2 \frac{\sum_{i \in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}.
\end{aligned}
$$

The first term, $(d_{A_k})^2 \frac{\sum_{i \in A_k} c_i(\sigma_i^{A_k})}{(\sigma^{A_k})^2}$, corresponds to the expected payoffs of members of alliance A_k. Similarly, terms of the form $(d_{A_{j+1}} - d_{A_j})^2 \frac{\sum_{i \in A_{j+1}} c_i(\sigma_i^{A_{j+1}})}{(\sigma^{A_{j+1}})^2}$ correspond to the expected payoffs of members in A_{j+1} who are not in A_j. The remaining terms capture the externalities induced by members of each alliance. For instance, the term $d_{A_k} \sum_{m=k}^{K} (d_{A_{m+1}} - d_{A_m}) \frac{2 \sum_{i \in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_m})^2}$ corresponds to the positive externalities (net of own payoffs) members of the first alliance A_k induce on members of all future alliances. Similarly, each term of the form $(d_{A_{j+1}} - d_{A_j}) \sum_{m=j+1}^{K} (d_{A_{m+1}} - d_{A_m}) \frac{2 \sum_{i \in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_m})^2}$ corresponds to the positive externalities (net of own payoffs) members of alliance A_j induce on future alliances. Naturally, the last alliance A_K has no alliances that follow it and therefore does not induce externalities of this sort.

Noticing the telescoping sum and setting $d_{A_{k-1}} = 0$, we can write

$$
W_k(0,0) = \sum_{m=k}^{K} \left((d_{A_m})^2 - (d_{A_{m-1}})^2 \right) \frac{\sum_{i \in A_m} c_i(\sigma_i^{A_m})}{(\sigma^{A_m})^2}.
$$

In particular, for $A_1 = N$, we get the formula asserted in the proposition. ■

4 Equilibrium and Social Planner Solutions for Exponential Costs

In this section, we focus on the specific class of exponential cost functions that underlie Figure 3 in the main text. Let

$$
c(\sigma) = e^{\sigma} = c_1(\sigma) = \beta_2 c_2(\sigma) = \beta_3 c_3(\sigma).
$$

It follows that:

$$c_1'(\sigma) = c(\sigma),\ c_2'(\sigma) = \frac{1}{\beta_2}c(\sigma)\ ,\ c_3'(\sigma) = \frac{1}{\beta_3}c(\sigma).$$

Observation OA 1. *If the alliance $\{1,2,3\}$ is active under the social planner's policy, the resulting total search scope is 6 and the total search cost is $\frac{3e^2}{\sqrt[3]{\beta_2\beta_3}}$.*

Indeed, search scopes are determined by the following system:

$$\begin{aligned}
\frac{2\left(c(\sigma_1) + \frac{1}{\beta_2}c(\sigma_2) + \frac{1}{\beta_3}c(\sigma_3)\right)}{\sigma_1+\sigma_2+\sigma_3} &= c(\sigma_1),\\
\frac{2\left(c(\sigma_1) + \frac{1}{\beta_2}c(\sigma_2) + \frac{1}{\beta_3}c(\sigma_3)\right)}{\sigma_1+\sigma_2+\sigma_3} &= \frac{1}{\beta_2}c(\sigma_2),\\
\frac{2\left(c(\sigma_1) + \frac{1}{\beta_2}c(\sigma_2) + \frac{1}{\beta_3}c(\sigma_3)\right)}{\sigma_1+\sigma_2+\sigma_3} &= \frac{1}{\beta_3}c(\sigma_3).
\end{aligned}$$

Solving the equations simultaneously we get:

$$\begin{aligned}
(\sigma_1+\sigma_2+\sigma_3) &= 6,\\
c(\sigma_1) + \frac{1}{\beta_2}c(\sigma_2) + \frac{1}{\beta_3}c(\sigma_3) &= \frac{3e^2}{\sqrt[3]{\beta_2\beta_3}}.
\end{aligned}$$

Observation OA 2. *If alliance $\{i,j\}$ is active under the social planner's policy, the resulting total search scope is 4 and the total search cost is $\frac{2e^2}{\sqrt[2]{\beta_i\beta_j}}$.*

In this case, search scopes are determined by the following system:

$$\begin{aligned}
\frac{2\left(\frac{1}{\beta_i}c(\sigma_i) + \frac{1}{\beta_j}c(\sigma_j)\right)}{\sigma_i+\sigma_j} &= c(\sigma_i)\frac{1}{\beta_i},\\
\frac{2\left(\frac{1}{\beta_i}c(\sigma_i) + \frac{1}{\beta_j}c(\sigma_j)\right)}{\sigma_i+\sigma_j} &= c(\sigma_j)\frac{1}{\beta_j}.
\end{aligned}$$

Solving the equations simultaneously we get, as stated:

$$\begin{aligned}
(\sigma_i+\sigma_j) &= 4,\\
\frac{1}{\beta_i}c(\sigma_i) + \frac{1}{\beta_j}c(\sigma_j) &= \frac{2e^2}{\sqrt[2]{\beta_i\beta_j}}.
\end{aligned}$$

Observation OA 3. *In equilibrium, in any alliance $A \subset \{1,2,3\}$, the total search scope is 2. Agents share the scope costs equally. Each agent's individual search cost is given by $\frac{1}{\beta_i}e^{\frac{2}{|A|}}$ and total cost is $\sum_{i \in A} \frac{1}{\beta_i}e^{\frac{2}{|A|}}$.*

To see this observation, observe that, in equilibrium, for any alliance A, individual search scopes are determined by:

$$\frac{2\frac{1}{\beta_i}c(\sigma_i^A)}{\sum_{i \in A} \sigma_i} = c(\sigma_i)\frac{1}{\beta_i} \quad \forall i \in A.$$

Solving this system yields:

$$\begin{aligned} \sum_{i \in A} \sigma_i &= 2, \\ \sigma_i &= \sigma_j \forall i \in A, \\ c(\sigma_i) &= \frac{1}{\beta_i}e^{\frac{2}{|A|}}. \end{aligned}$$

Straightforward calculations then generate the following two observations, where $\{1\}\{2\}\{3\}$ implies that agent 1 leaves before agent 2, who leaves before agent 3; $\{1,2\}\{3\}$ implies that agents 1 and 2 form an exit wave and leave first, followed by agent 3; and so on.

Observation OA 4. *There are four patterns of exit waves that can happen in equilibrium:*

1. $\{1\}\{2\}\{3\}$, which requires $\beta_2 > e^{\frac{1}{3}}$ and $\frac{\beta_3}{\beta_2} > e$;
2. $\{1,2\}\{3\}$, which requires $\beta_2 < e^{\frac{1}{3}}$ and $\beta_3 > e^{\frac{4}{3}}$;
3. $\{1\}\{2,3\}$, which requires $\beta_2 > \frac{4}{3}$ and $\frac{\beta_3}{\beta_2} < e$;
4. $\{1,2,3\}$, which requires $\beta_2 < \frac{4}{3}$ and $\beta_3 < e^{\frac{4}{3}}$.

Observation OA 5. *There are four patterns of exit waves that can happen in social planner's solution:*

1. $\{1\}\{2\}\{3\}$, which requires $\min\left(\frac{2}{9}\frac{1}{\sqrt[3]{\beta_2\beta_3}}, \frac{1}{2}\frac{1}{\sqrt[2]{\beta_2\beta_3}}\right) > \frac{1}{\beta_3}$ and $\frac{1}{\beta_3} > \frac{1}{\sqrt[2]{\beta_2\beta_3}} - \frac{1}{3}\frac{1}{\sqrt[3]{\beta_2\beta_3}}$;
2. $\{1,2\}\{3\}$, which requires $\min\left(\frac{2}{9}\frac{1}{\sqrt[3]{\beta_2\beta_3}}, \frac{1}{2}\frac{1}{\sqrt[2]{\beta_2\beta_3}}\right) > \frac{1}{\beta_3}$ and $\frac{1}{\beta_3} < \frac{1}{\sqrt[2]{\beta_2\beta_3}} - \frac{1}{3}\frac{1}{\sqrt[3]{\beta_2\beta_3}}$;
3. $\{1\}\{2,3\}$, which requires $\frac{1}{2}\frac{1}{\sqrt[2]{\beta_2\beta_3}} < \min\left(\frac{2}{9}\frac{1}{\sqrt[3]{\beta_2\beta_3}}, \frac{1}{\beta_3}\right)$;
4. $\{1,2,3\}$, which requires $\frac{2}{9}\frac{1}{\sqrt[3]{\beta_2\beta_3}} < \min\left(\frac{1}{2}\frac{1}{\sqrt[2]{\beta_2\beta_3}}, \frac{1}{\beta_3}\right)$.

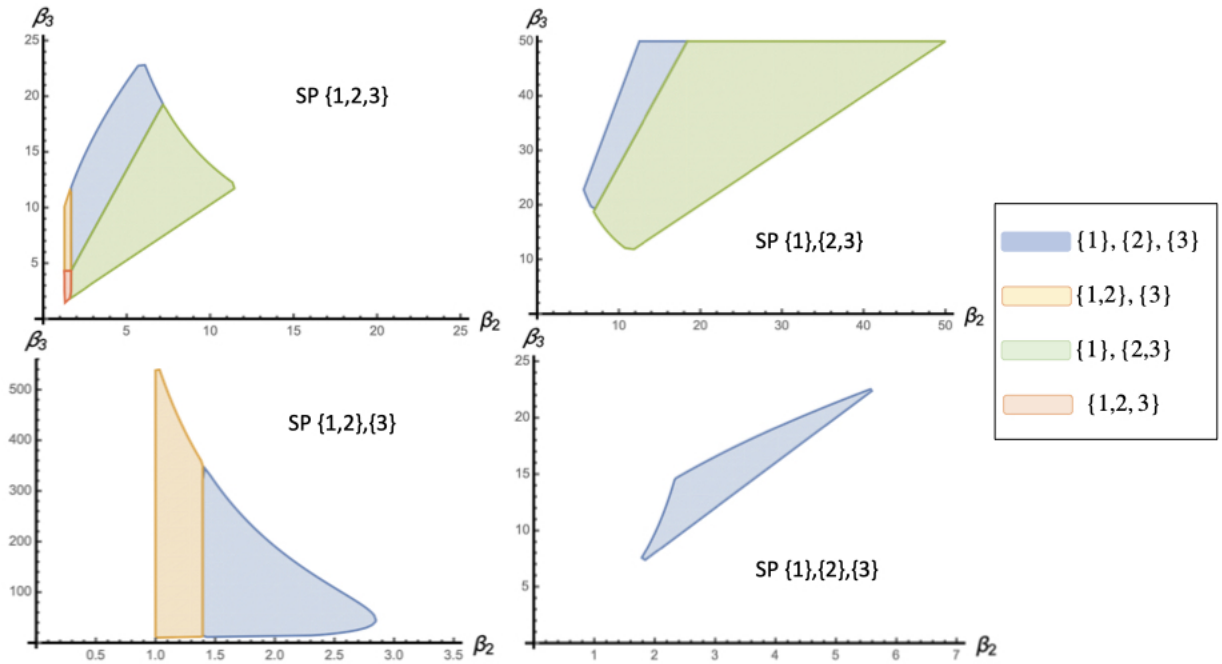

Figure 1: Comparison of Equilibrium and Social Planner Solutions

Figure 1 here expands on Figure 3 in the main text and illustrates the wedge between the equilibrium and social exit wave sequences. As can be seen, while the "weak order" by which exit waves occur in equilibrium is preserved by the social planner—see Lemma 1 in the main text—the pattern of exit waves may still differ dramatically depending on whether agents have full discretion or are governed by the socially optimal policy.

5 Penalties for Later Innovations

We now consider an extension of our model where stopping earlier grants one an advantage. For simplicity, we consider here a team of only two agents. We assume that the first agent to stop, say at time t, receives M_t. The second agent to stop, say at time $s > t$, receives αM_s, with $\alpha < 1$. If both agents stop at the same time t, they both receive M_t.[1]

Our first proposition shows that if the stopping times are (weakly) ordered at any point, they are weakly ordered everywhere. Thus, the order of exits is deterministic.

[1]The analysis naturally extends to N agents via a decreasing sequence of discounts: $\alpha_0 = 1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$. In addition, one could consider more continuous version of this setup, where the second agent who stops at time $s > t$ receives $M_t + \alpha(M_s - M_t)$. That model generates qualitatively similar results, but is more cumbersome to analyze.

Proposition OA 2. *If $g_i^{\{1,2\}}(M^*) \geq g_j^{\{1,2\}}(M^*)$ for some M^*, then $g_i^{\{1,2\}}(M) \geq g_j^{\{1,2\}}(M)$ for all M.*

Proof. If $g_i^{\{1,2\}}(M^*) \geq g_j^{\{1,2\}}(M^*)$ then it must be the case that $V_j^j(M^*, g_i^T(M^*)) \geq M^*$ and $V_i^{\{1,2\}}(M, g_i^T(M^*)) = V_i^i(M^*, g_i^{\{1,2\}}(M^*)) = M^*$. This implies that

$$g_i^{\{1,2\}}(M^*) - M^* + \frac{(\sigma_i^i)^2}{2c_i(\sigma_i^i)} \leq 0$$
$$\alpha M + \frac{c_j(\sigma_j^j)}{(\sigma_j^j)^2}\left(g_i^{\{1,2\}}(M^*) - M^* + \frac{\alpha(\sigma_j^j)^2}{2c_j(\sigma_j^j)}\right)^2 \geq M.$$

Now, since $\alpha \leq 1$, the second inequality implies that

$$\alpha M + \frac{c_j(\sigma_j^j)}{(\sigma_j^j)^2}\left(g_i^{\{1,2\}}(M^*) - M^* + \frac{\alpha(\sigma_j^j)^2}{2c_j(\sigma_j^j)}\right)^2 \geq M$$
$$g_i^{\{1,2\}}(M^*) - M^* + \frac{\alpha(\sigma_j^j)^2}{2c_j(\sigma_j^j)} \geq 0$$
$$g_i^{\{1,2\}}(M^*) - M^* + \frac{(\sigma_j^j)^2}{2c_j(\sigma_j^j)} \geq 0.$$

From this system, it must be the case that $\frac{(\sigma_j^j)^2}{2c_i(\sigma_j^j)} \geq \frac{(\sigma_i^i)^2}{2c_i(\sigma_i^i)}$. Now, towards a contradiction, suppose there exists a different M such that $g_i^{\{1,2\}}(M) < g_j^{\{1,2\}}(M)$. Then, $V_j^j(M, g_i^{\{1,2\}}(M)) = M$ and $V_i^i(M, g_i^{\{1,2\}}(M)) > M$, which yield the following inequalities:

$$g_i^{\{1,2\}}(M) - M + \frac{(\sigma_i^j)^2}{2c_i(\sigma_i^i)} > 0 \qquad \textit{and}$$
$$g_i^{\{1,2\}}(M) - M + \frac{(\sigma_j^j)^2}{2c_j(\sigma_j^j)} \leq 0.$$

These can hold only if $\frac{(\sigma_j^j)^2}{2c_j(\sigma_j^j)} < \frac{(\sigma_i^j)^2}{2c_i(\sigma_i^i)}$, generating a contradiction. ■

In general, there is a *leader*—the agent who exits early—and a *follower*—the agent who exits later. As we now show, the leader's stopping boundary remains her equilibrium stopping boundary regardless of α and is governed by the drawdown identified in Proposition 2 in the text. Both the leader and the follower's search scopes, when searching together or separately, also follow identical considerations to those pertaining to the setting analyzed in the paper and described in Proposition 1 of the text. In contrast, the follower's stopping boundary does change since her rewards are scaled down by α.

We will call i is the leader if

$$\frac{(\sigma_i^{\{1,2\}}+\sigma_j^{\{1,2\}})^2}{2(c_i(\sigma_i^{\{1,2\}}))} \leq \frac{(\sigma_i^{\{1,2\}}+\sigma_j^{\{1,2\}})^2}{2(c_j(\sigma_j^{\{1,2\}}))}.$$

Henceforth we will use L, F to denote respectively the leader and the follower.

5.1 Equilibrium Scopes

The proof of Proposition 1 does not hinge on agents receiving the full value of the observed maximum.[2] Therefore, analogous analysis yields that the search scopes (when interior) satisfy the following:

$$\frac{2c_L(\sigma_L^{\{1,2\}})}{c'_L(\sigma_L^{\{1,2\}})} = \frac{2c_F(\sigma_F^{\{1,2\}})}{c'_F(\sigma_F^{\{1,2\}})} = \sigma^{\{1,2\}}.$$

5.2 Leader's Optimal Stopping

Since the search scope in any alliance is constant, the leader's optimal stopping boundary satisfies

$$g_L^{\{1,2\}}(M) = M - \frac{(\sigma_L^{\{1,2\}}+\sigma_F^{\{1,2\}})^2}{2(c_L(\sigma_L^{\{1,2\}}))}.$$

5.3 Follower's Optimal Stopping

Urgun and Yariv (2021b) show that, in the solo-search problem, the optimal scope is independent of α and thus identical to that identified in our baseline model. Their analysis also shows that the follower's optimal stopping boundary is given by

$$g_F^F(M) = M - \frac{\alpha(\sigma_F^F)^2}{2c_F(\sigma_F^F)}.$$

Since the search scopes of the two agents differ, it is possible that $g_L^{\{1,2\}}(M) \geq g_F^F(M)$. In that case, the follower stops at the same time as the leader.

If $g_L^{\{1,2\}}(M) \leq g_F^F(M)$, when the leader stops, the follower's payoff is at least as high as that derived from stopping immediately and receiving M. Again, utilizing Urgun and Yariv

[2] See also Urgun and Yariv (2021b) for additional details on the optimal policy pertaining to scaled-down search rewards.

(2021b), we know that the continuation payoff would be

$$V_F^F(M, g_L^{\{1,2\}}) = \alpha M + \frac{c_F(\sigma_F^F)}{(\sigma_F^F)^2}(g_L^{\{1,2\}}(M) - g_F^F(M))^2,$$

whereas stopping would yield an immediate M. Since both stopping boundaries are drawdown stopping boundaries, $\frac{c_F(\sigma_F^F)}{(\sigma_F^F)^2}(g_L^{\{1,2\}}(M) - g_F^F(M))^2$ is a constant independent of M. When $(1-\alpha)M$ is larger than this constant, the follower stops as soon as the leader does.

6 Fixed Search Scope

Suppose the search scope is fixed and determined at the outset. Agents cannot then adjust their search scope as alliances change in size or composition. To illustrate the impacts of such limited adjustment possibilities, we consider the simple case of two agents with well-ordered costs. That is, agent 1's search cost is given by $c(\cdot)$, while agent 2's search cost is given by $\beta c(\cdot)$ with $\beta > 1$. In this case, agents' optimal solo search scopes coincide. We consider two cases: one in which both agents are restricted to a search scope that is optimal for their single-agent search; the second where both agents are restricted to a search scope that is optimal when they search jointly.

6.1 Single-agent Search Scope

Suppose agents are restricted to the search scope σ^I that would be optimal were they each searching individually on their own:

$$\frac{2c(\sigma^I)}{c'(\sigma^I)} = \sigma^I.$$

As we soon show, this is, in fact, the welfare-maximizing fixed search scope.

When searching as a team, the optimal search scope for each individual would be given by $\sigma^T < \sigma^I$, where

$$\frac{c(\sigma^T)}{c'(\sigma^T)} = \sigma^T.$$

The corresponding drawdown for the team is then be given by

$$d^T = \frac{2(\sigma^T)^2}{c(\sigma^T)}.$$

In contrast, if both agents are restricted to using σ^I, the equilibrium drawdown of the initial alliance consisting of both agents is given by

$$d^T_{restricted} = \frac{2(\sigma^I)^2}{c(\sigma^I)}.$$

Claim OA 1. *$d^T_{restricted} \leq d^T$.*

Proof. Recall that σ^T minimizes $\frac{c(\sigma)}{(2\sigma)^2}$. Inverting the ratio implies that we must have $\frac{4(\sigma^I)^2}{2c(\sigma^I)} \leq \frac{4(\sigma^T)^2}{2c(\sigma^T)}$. ■

If agent 2 continues searching after agent 1 exits, the search scope for agent 2 is optimal, and her solo drawdown is therefore unaffected by the constraint and given by Proposition 2 in the text—call this drawdown d^I. In particular, if the two agents leave at disjoint times in the unrestricted case, so that $d^T < d^I$, the claim implies that the agents would also leave at disparate times in the restricted case, since $d^T_{restricted} \leq d^T < d^I$. However, if agents depart jointly in the unrestricted case, that exit wave might disappear in the restricted case. To see this, observe that in order to have an exit wave in the unrestricted case we need

$$\frac{4(\sigma^T)^2}{c(\sigma^T)} \geq \frac{\beta(\sigma^I)^2}{c(\sigma^I)}.$$

For this wave to break in the restricted case we need

$$\frac{4(\sigma^I)^2}{c(\sigma^I)} \leq \frac{\beta(\sigma^I)^2}{c(\sigma^I)}.$$

If $\beta \geq 4$, the restricted case will have agents departing at separate points, regardless of the structure of exit waves in the unrestricted case. To see the change in individual welfare, consider an agent utilizing a drawdown stopping boundary with drawdown size d and search scope σ that comes at a cost $c(\sigma)$. From Urgun and Yariv (2021b), the expected value for that agent is given by

$$V(d, \sigma) = d - \frac{d^2}{4\sigma^2} c(\sigma).$$

Therefore, in the unrestricted case, the expected value for agent 1, who is the first to leave is given by

$$V^1 = \frac{(\sigma^T)^2}{c(\sigma^T)} = \frac{d^T}{2},$$

whereas, in the restricted case, the expected welfare for agent 1 is given by

$$V^1_{restricted} = \frac{(\sigma^I)^2}{c(\sigma^I)} = \frac{d^T_{restricted}}{2}.$$

In particular, from the claim above, the expected welfare of agent 1 necessarily decreases.

The expected welfare of agent 2 depends on whether she departs when agent 1 does or continues searching. We omit its derivations. In either case, however, the restriction on the search scope leads to a decrease in her expected welfare as well.

6.2 Team Search Scope

We now consider the case in which agents are restricted to the optimal search scope for the team, σ^T defined above.

Our analysis so far implies that agent 1's expected welfare is unaffected by this constraint and, using our previous notation, given by $d^T/2$.

Agent 2 may be affected if she continues searching after agent 1 terminates her search. Indeed, when agent 2 can adjust her search scope to its optimal solo-level of σ^I, the corresponding optimal drawdown is given by:

$$d^I = \frac{\beta(\sigma^I)^2}{2c(\sigma^I)}.$$

However, when agent 2 is restricted to continue searching with scope σ^T, she accordingly adjusts her drawdown to

$$d^I_{restricted} = \frac{\beta(\sigma^T)^2}{2c(\sigma^T)}.$$

Claim OA 2. $d^I_{restricted} \leq d^I$.

Proof. Recall that the optimal solo search scope σ^I minimizes $\frac{c(\sigma)}{\sigma^2}$. Inverting the ratio immediately yields the claim as in the previous proof. ■

Intuitively, since agent 2 is now constrained to search with a less efficient scope, she responds by searching for a shorter duration. As before, if the unrestricted setting generates an exit wave with both agents ceasing search in unison, $d^T \geq d^2$ and the restricted environment would yield the same exit wave. However, if the unrestricted search leads the agents to depart at different points, this need not be the case when search scope is restricted to stay constant. Indeed, the two agents would terminate their search together if

$$\frac{4(\sigma^T)^2}{c(\sigma^T)} \geq \frac{\beta(\sigma^T)^2}{c(\sigma^T)}.$$

Again, $\beta = 4$ is the critical value: when $\beta < 4$ so that agents' costs are sufficiently similar, a non-trivial exit wave may occur.

We can now assess the welfare loss of agent 2. Certainly, if the agents depart together both in the restricted and the unrestricted setting, the constraint on scope has no bite and neither agent's expected welfare is affected.

In general, agent 1 departs with a maximal observed value of M when the search observation hits precisely $X = M - d^T$. If agent 2 continues searching on her own in the unrestricted case, her resulting continuation value is

$$V^2(M, M - d^T) = M + \frac{c(\sigma^I)}{\beta(\sigma^I)^2}\left(d^T - \frac{\beta(\sigma^I)^2}{2c(\sigma^I)}\right)^2.$$

When restricted to continue searching with a scope of σ^T, agent 2's continuation value is given by

$$V^2_{restricted}(M, M - d^T) = M + \frac{c(\sigma^T)}{\beta(\sigma^T)^2}\left(d^T - \frac{\beta(\sigma^T)^2}{2c(\sigma^T)}\right)^2.$$

The difference captures the welfare loss. In particular, whenever there is a non-trivial exit wave in the restricted setting, when agents depart in sequence in the unrestricted setting, the second term of $V^2(M, M - d^T)$ captures the loss in welfare.

6.3 Optimal Fixed Search Scope

One could tailor the fixed search scope to maximize the maximal welfare of the team. For illustration, suppose the costs of the agents are close enough so that a non-trivial exit wave occurs when the search scope is fixed. Namely, suppose that $\beta \leq 4$. For any fixed search scope $\tilde{\sigma}$, the expected welfare of agent 1 is given by

$$\max_d V^1(d, \tilde{\sigma}) = d - \frac{d^2}{(2\tilde{\sigma})^2} c(\tilde{\sigma}).$$

Agent 1 then chooses her optimal drawdown size $\tilde{d}$ to satisfy the resulting first-order condition:

$$\frac{\partial V_1}{\partial d} = 1 - 2\tilde{d}\frac{c(\tilde{\sigma})}{(2\tilde{\sigma})^2} = 0 \Rightarrow \tilde{d} = \frac{2\tilde{\sigma}^2}{c(\tilde{\sigma})}.$$

This yields an expected welfare of

$$V^1_{restricted}(\tilde{d}, \tilde{\sigma}) = \frac{2\tilde{\sigma}^2}{c(\tilde{\sigma})} - \frac{\left(\frac{2\tilde{\sigma}^2}{c(\tilde{\sigma})}\right)^2}{(2\tilde{\sigma})^2} c(\tilde{\sigma}^2) = \frac{\tilde{\sigma}^2}{c(\tilde{\sigma})}.$$

Since $\beta \leq 4$, agent 2 exits at the same time and her expected welfare can be calculated similarly:

$$V^2_{restricted}(\tilde{d}, \tilde{\sigma}) = \frac{2\tilde{\sigma}^2}{c(\tilde{\sigma})} - \frac{\left(\frac{2\tilde{\sigma}^2}{c(\tilde{\sigma})}\right)^2}{(2\tilde{\sigma})^2} c(\tilde{\sigma}^2)\frac{1}{\beta} = \left(2 - \frac{1}{\beta}\right)\frac{\tilde{\sigma}^2}{c(\tilde{\sigma})}.$$

The resulting expected welfare $W(\tilde{\sigma})$ for both agents is then:

$$\begin{aligned} W_{restricted}(\tilde{\sigma}) &= V^1_{restricted}(\tilde{d}, \tilde{\sigma}) + V^2_{restricted}(\tilde{d}, \tilde{\sigma}) \\ &= \left(3 - \frac{1}{\beta}\right)\frac{\tilde{\sigma}^2}{c(\tilde{\sigma})}. \end{aligned}$$

The overall welfare is then maximized when $\frac{\tilde{\sigma}^2}{c(\tilde{\sigma})}$ is maximized. Now, recall that the optimal individual search scope is defined by $\sigma^I = \arg\min_{\tilde{\sigma}} \frac{c(\tilde{\sigma})}{\tilde{\sigma}^2}$. In particular, the optimal fixed search scope is the individually-optimal search scope σ^I, as analyzed in Section 6.1 above.

7 General Search Technologies

In this section, we consider general search technologies. Let f^A denote the search technology for alliance A, where $f : [\overline{\sigma},\ \underline{\sigma}]^{|A|} \rightarrow [\underline{k}, \overline{K}]$ with $\underline{k} > 0$ and $\overline{K} < \infty$. We assume f^A is continuous, increasing, and sufficiently smooth.

We show that the only direct impact of richer search technologies is through the induced search scopes. Once search scopes are adjusted, the stopping rules follow those identified in the main text.

We directly write the HJBs and the relevant first-order conditions, which are derived similarly to those in the main text.

For any agent i, the single-agent HJB equation is given by

$$\sup_{\sigma_i}\{\frac{1}{2}(f^i(\sigma_i^i)^2\frac{\partial^2 V(M,X)}{\partial X^2} - c_i(\sigma_i^i)\} = 0.$$

The first-order condition can be written as

$$\frac{2\frac{\partial f^i(\sigma_i^i)}{\partial \sigma_i^i}c_i(\sigma_i^i)}{f^i(\sigma_i^i)} - c_i'(\sigma_i^i) = 0.$$

After rearranging,

$$\frac{2\frac{\partial f^i(\sigma_i^i)}{\partial \sigma_i^i}c_i(\sigma_i^i)}{c_i'(\sigma_i^i)} = f^i(\sigma_i^i).$$

Similarly, in any active alliance A, we can reach the following first-order condition:

$$\frac{2\frac{\partial f^A(\sigma^A)}{\partial \sigma_i^i}c_i(\sigma_i^A)}{c_i'(\sigma_i^A)} = f^A(\sigma^A).$$

Once search scopes are identified, similar analysis to that appearing in the main text yields optimal drawdown stopping boundaries. The drawdown size of an alliance A is given by

$$\min_{i\in A}\frac{f^A(\sigma^A)}{2c_i(\sigma_i^A)}.$$

8 Independent Samples

In order to analyze the case with independent search outcomes, we turn our analysis into a discrete-time setup with two agents, 1 and 2. As in our benchmark setting, we retain the normality of observations assumptions. We also continue to assume that agents jointly control the variance. If an alliance $A \subset \{1,2\}, A \neq \emptyset$ is conducting search at time t, their observation is drawn from a normal distribution with mean 0 and standard deviation $\sigma_t^A = \sum_{i\in A}\sigma_{i,t}^A$. The cost is still paid per draw, but now accumulates in discrete periods as opposed to continuous time. We maintain our assumptions on the cost of search scope, as well as the description of ultimate search rewards, which follow from the maximal observation during search. When search is conducted by a single agent, Urgun and Yariv (2021a) show that the optimal stopping policy is characterized by a threshold and the optimal search scope is constant. They show that:

Proposition OA 3 (Urgun and Yariv, 2021a)**.** *For a given search scope σ, it is optimal to stop once the satisficing threshold $S_i^i(\sigma)$ is reached, where $S_i^i(\sigma)$ solves*

$$c(\sigma) = \int_{S_i^i(\sigma)}^{\infty} (x - S_i^i(\sigma))\phi^{\sigma}(x)dx.$$

The optimal search scope σ_i maximizes $\psi^{-1}\left(\frac{c(\sigma)}{\sigma}\right)\sigma$ where $\psi(v) = \phi^1(v) - v \times (1 - \Phi^1(v))$ and ϕ^{σ} denotes the normal probability density function with mean 0 and standard deviation σ. The payoff from optimal search is $S_i^i(\sigma_i)$.

As noted in Urgun and Yariv (2021a), the function $\psi(v)$ and its inverse are difficult to simplify further in terms of elementary functions for closed-form characterizations, but they can be easily tabulated and some properties of both $\psi(v)$ and its inverse are well known.

Once one of the players departs, the optimal search is stationary and identified by the proposition above. Moreover, similar arguments to those presented in the text can be used to show that equilibrium search scopes are constant within any alliance. Let $\sigma_i^{\{1,2\}}$ for $i \in \{1,2\}$ denote the search scope of agent i when both agents search jointly. There are two thresholds $S_1^{\{1,2\}}(\sigma^{\{1,2\}})$ and $S_2^{\{1,2\}}(\sigma^{\{1,2\}})$ that determine when agent 1 and agent 2 cease search, respectively. Equilibrium thresholds equal the continuation values given the search scopes, while equilibrium search scopes are optimal given the thresholds.

For given search scopes, the optimal thresholds must satisfy, for $i = 1, 2$:

$$\begin{aligned}
S_i^{\{1,2\}}(\sigma^{\{1,2\}}) &= -c_i(\sigma_i^{\{1,2\}}) + \int_{-\infty}^{\min\{S_1^{\{1,2\}}(\sigma^{\{1,2\}}),S_2^{\{1,2\}}(\sigma^{\{1,2\}})\}} S_i^{\{1,2\}}(\sigma^{1,2})\phi^{\sigma^{\{1,2\}}}(x)dx \\
&+ \int_{\min\{S_1^{\{1,2\}}(\sigma^{\{1,2\}}),S_2^{\{1,2\}}(\sigma^{\{1,2\}})\}}^{\max\{S_1^{\{1,2\}}(\sigma^{\{1,2\}}),S_2^{\{1,2\}}(\sigma^{\{1,2\}})\}} \Big[I\left(S_i^{\{1,2\}}(\sigma^{\{1,2\}}) \geq S_j^{\{1,2\}}(\sigma^{\{1,2\}})\right) S_i^i(\sigma_i) \\
&+ I\left(S_i^{\{1,2\}}(\sigma^{\{1,2\}}) \leq S_j^{\{1,2\}}(\sigma^{\{1,2\}})\right) x \Big] \phi^{\sigma^{\{1,2\}}}(x)dx \\
&+ \int_{\max\{S_1^{\{1,2\}}(\sigma^{\{1,2\}}),S_2^{\{1,2\}}(\sigma^{\{1,2\}})\}}^{\infty} x\phi^{\sigma^{\{1,2\}}}(x)dx.
\end{aligned}$$

While amenable to numerical analysis, a closed-form description of the optimal thresholds is naturally challenging to derive in this setting. Nonetheless, one can readily see one qualitative distinction between equilibrium outcomes in this setting and those we identify in the paper—the sequence of exit waves is no longer deterministic. Indeed, when sufficiently extreme observations occur, both agents may leave at once. However, for moderately high observations, one agent may leave on her own.

The equilibrium search scopes are chosen so that

$$\sigma_i^{\{1,2\}} = \arg\max S_i^{\{1,2\}}(\sigma^{\{1,2\}}) \text{ for } i = 1, 2.$$

Additional players introduce further hurdles to tractability since, for each alliance, one needs to consider the threshold corresponding to each agent. We leave the complete analysis of the independent case for future research.

References

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