Information Acquisition in Committees: Technical Addendum

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1 Proof of Proposition 3

Consider a committee of size n. We look for the optimal mechanism under the restriction that all n players acquire information.

The problem is:

$$\max_{\gamma(0),\ldots,\gamma(n)\in[0,1]} -(1-q)\,P(G)+\sum_{k=0}^{n} v(k)\,\gamma(k)$$
$$\text{s.t.}\ \sum_{k=0}^{n} a(k)\,\gamma(k)\geqslant c$$
$$\sum_{k=0}^{n} b(k)\,\gamma(k)\geqslant c,$$

where

$$v(k)=\binom{n}{k}f(k,n),$$

$$a(k)=\binom{n-1}{k-1}f(k,n)-\binom{n-1}{k}f(k+1,n)$$

is the coefficient of $\gamma(k)$ in $IC(i)$, and

$$b(k)=\binom{n-1}{k}f(k,n)-\binom{n-1}{k-1}f(k-1,n)$$

is the coefficient of $\gamma(k)$ in $IC(g)$. We use the convention $\binom{n-1}{-1}=\binom{n-1}{n}=0$.

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Our optimization problem falls under the class of problems known as parametric linear programs. In particular, notice that the solution is continuous in the cost c (see, for instance, Zhang and Liu [1990]).

The goal is to show that when p is sufficiently close to one the optimal mechanism takes the form:

$$\bar{\gamma}_n(0) = \ldots = \bar{\gamma}_n\left(\hat{k}-1\right) = 0, \quad \bar{\gamma}_n\left(\hat{k}\right) = \alpha, \quad \bar{\gamma}_n\left(\hat{k}+1\right) = \ldots = \bar{\gamma}_n(k_n - 1) = 1,$$
$$\bar{\gamma}_n(k_n) = \ldots = \bar{\gamma}_n\left(\bar{k}-1\right) = 0, \quad \bar{\gamma}_n\left(\bar{k}\right) = \beta, \quad \bar{\gamma}_n\left(\bar{k}+1\right) = \ldots = \bar{\gamma}_n(n) = 1,$$

where $\alpha, \beta \in [0,1]$, and $0 < \hat{k} < k_n \leqslant \bar{k} < n$.

We assume that p is sufficiently large. Of course, $a(0) = -f(1,n) > 0$, $a(n) = f(n,n) > 0$, $b(0) = f(0,n) < 0$ and $b(n) = -f(n-1,n) < 0$.

Notice that for $k = 1, \ldots, n-1$, we can rewrite $a(k)$ and $b(k)$ as

$$a(k) = \binom{n-1}{k-1}\frac{1}{k}\left[(n(1-p)-k)\,qP(I)(1-p)^k p^{n-k-1} + (k-np)(1-q)P(G)p^k(1-p)^{n-k-1}\right],$$

$$b(k) = \binom{n-1}{k-1}\frac{1}{k}\left[(k-n(1-p))\,qP(I)(1-p)^{k-1} p^{n-k} + (np-k)(1-q)P(G)p^{k-1}(1-p)^{n-k}\right].$$

Clearly, $a(k) < 0$ and $b(k) > 0$ for any $k \in [n(1-p), np]$.

Since p is close to one, $n(1-p) < 1$ and $np > n-1$ and, therefore, $a(k) < 0$ and $b(k) > 0$ for every $k = 1, \ldots, n-1$.

Throughout, we assume that n is odd and that $k_n = k_{n-1}$ (so that $IC(i)$ is the first constraint to bind when the device is Bayesian). In this case, k_n is equal to $\frac{n+1}{2}$.[1]

We know that when the cost is $\hat{c} = \binom{n-1}{k_n-1} f(k_n, n)$, the Bayesian device satisfies the $IC(i)$ constraint with equality. For costs above $\hat{c}$ we need to introduce distortions in order to induce all n players to acquire information. We also know from Proposition 3* in the Appendix of the paper that for c sufficiently close to $\hat{c}$ it is optimal to distort the mechanism at $k_n = \frac{n+1}{2}$ and set $\gamma(k_n)$ smaller than one. As c increases, $\gamma(k_n)$ decreases. Notice, however, that there exists a critical value of the cost $\bar{c} > \hat{c}$ such that at $\bar{c}$ the optimal mechanism is $\gamma(0) = \ldots = \gamma\left(\frac{n-1}{2}\right) = 0$, $\gamma\left(\frac{n+1}{2}\right) \in (0,1)$, $\gamma\left(\frac{n+3}{2}\right) = \ldots = \gamma(n) = 1$, and the value of $\gamma\left(\frac{n+1}{2}\right)$ is such that both constraints are satisfied with equality. To see this, note that if $\gamma(0) = \ldots = \gamma\left(\frac{n+1}{2}\right) = 0$ and $\gamma\left(\frac{n+3}{2}\right) = \ldots = \gamma(n) = 1$, then the LHS of the $IC(g)$ constraint is equal to $-\binom{n-1}{k_n} f(k_n; n) < 0$.

We now show that as the cost increases above $\bar{c}$ it is optimal to continue decreasing the value of $\gamma\left(\frac{n+1}{2}\right)$ and to start increasing the value of $\gamma\left(\frac{n-1}{2}\right)$. More generally, we prove the following.

[1] The cases in which n is even and/or $k_n = k_{n-1} + 1$ can be analyzed in a similar way.

Claim 1 *Assume that we are at a point $c > \bar{c}$ where the optimal mechanism is*

$$\begin{aligned}
&\bar{\gamma}_n(0) = \ldots = \bar{\gamma}_n\left(\hat{k}\right) = 0, && && \bar{\gamma}_n\left(\hat{k}+1\right) = \ldots = \bar{\gamma}_n(k_n - 1) = 1, \\
&\bar{\gamma}_n(k_n) = \ldots = \bar{\gamma}_n\left(\bar{k}-1\right) = 0, && \bar{\gamma}_n\left(\bar{k}\right) = \beta, && \bar{\gamma}_n\left(\bar{k}+1\right) = \ldots = \bar{\gamma}_n(n) = 1,
\end{aligned} \tag{1}$$

$\beta \in (0,1)$, and $0 < \hat{k} < k_n \leqslant \bar{k} < n$. Suppose that the cost increases. Then it is optimal to continue decreasing $\bar{\gamma}_n\left(\bar{k}\right)$ and to start increasing $\bar{\gamma}_n\left(\hat{k}\right)$.

In what follows, we provide a proof for Claim 1. A symmetric claim also holds:

Claim 2 *Assume that we are at a cost $c > \bar{c}$ where the optimal mechanism is*

$$\begin{aligned}
&\bar{\gamma}_n(0) = \ldots = \bar{\gamma}_n\left(\hat{k}-1\right) = 0, && \bar{\gamma}_n\left(\hat{k}\right) = \alpha, && \bar{\gamma}_n\left(\hat{k}+1\right) = \ldots = \bar{\gamma}_n(k_n - 1) = 1, \\
&\bar{\gamma}_n(k_n) = \ldots = \bar{\gamma}_n\left(\bar{k}-1\right) = 0, && && \bar{\gamma}_n\left(\bar{k}\right) = \bar{\gamma}_n\left(\bar{k}+1\right) = \ldots = \bar{\gamma}_n(n) = 1,
\end{aligned}$$

where $\alpha \in (0,1)$, and $0 < \hat{k} < k_n \leqslant \bar{k} < n$. Suppose that the cost increases. Then it is optimal to continue increasing $\bar{\gamma}_n\left(\hat{k}\right)$ and to start decreasing $\bar{\gamma}_n\left(\bar{k}\right)$.

The proof of Claim 2 is identical to that of Claim 1 and is thus omitted. The combination of these two claims (together with Remark 3 below) provide the proof of Proposition 3.[2]

Proof of Claim 1

Note that the optimal device is the solution to a linear programming problem with two constraints, $IC(i)$ and $IC(g)$, and the additional constraints that every $\gamma(k)$ belongs to $[0,1]$. It follows that there will be at most two values of k at which $\gamma(k)$ is different from 0 or 1 (see, e.g., Luenberger [1965], Chapter 3). Clearly, the optimal mechanism is continuous in c. Thus, if we start from the device (1) and increase c by a small amount, the optimal mechanism is such that the value of $\bar{\gamma}_n\left(\bar{k}\right)$ is close to β. Therefore, if we start from (1) and increase c, one change must pertain to $\bar{\gamma}_n\left(\bar{k}\right)$.

In principle, there are different ways to satisfy the constraints when c increases:

1. Decrease the value of $\gamma\left(\bar{k}\right)$ and increase the value of $\gamma(k)$ for some $k = 1, \ldots, \hat{k}$;

2. Decrease the value of $\gamma\left(\bar{k}\right)$ and increase the value of $\gamma(k)$ for some $k = k_n \left(= \frac{n+1}{2}\right), \ldots, \bar{k} - 1$;

3. Increase the value of $\gamma\left(\bar{k}\right)$ and decrease the value of $\gamma(k)$ for some $k = \bar{k} + 1, \ldots, n - 1$;

[2]Note that in generic environments the optimal distortionary device entails randomization for at least one profile of reports. Our proof does, however, extend to non-generic cases in which for some cost levels, the optimal distortionary device entails no randomization.

4. Increase the value of $\gamma\left(\bar{k}\right)$ and decrease the value of $\gamma\left(k\right)$ for some $k = \hat{k} + 1, \ldots, k_n - 1 \left(= \frac{n-1}{2}\right);$

5. Increase the value of $\gamma\left(\bar{k}\right)$ and increase the value of $\gamma\left(0\right);$

6. Decrease the value of $\gamma\left(\bar{k}\right)$ and decrease the value of $\gamma\left(n\right).$

In all cases, the optimal thing to do is to satisfy both constraints with equality. Recall that we start at a point where both constraints are binding and the mechanism is not Bayesian. If we end up with a mechanism under which one constraint is not binding, the mechanism cannot be optimal.[3]

Below we prove the following facts:

A In case 1, the optimal distortion is to use $\hat{k}$, the largest k available.

B Any change in which we increase $\gamma(k')$ and decrease $\gamma(k'')$, where $k' = k_n, ..., n-2$ and $k'' = k'+1, ..., n-1$ has a negative effect on the designer's expected utility (the objective function). Furthermore, this change is worse than any change in which we decrease $\gamma(k')$ and increase $\gamma(k)$, where $k = 1, ..., \hat{k}$.

C Case 4 is not feasible.

D Case 5 is not feasible.

E Case 6 is not feasible.

Note that the distortions mentioned in Fact A certainly generate a decrease in the expected value of the designer's objective function. Fact B implies that case 3 cannot be optimal directly. In fact, it implies that distortions of the type specified in case 3 generate lower expected values to the designer than distortions of the type specified in case 1. In particular, the former yield a decrease in the designer's expected value as well. Fact B also implies that case 2 cannot be optimal. Indeed, suppose we end up with a device in which $\gamma\left(\bar{k}\right) \in (0,1)$ and $\gamma\left(k\right) \in (0,1)$ for some $k = k_n, \ldots, \bar{k}-1$. Then consider the following deviation. Decrease the value of $\gamma\left(k\right)$ and increase the value of $\gamma\left(\bar{k}\right)$ so that the LHS of both constraints decreases by the same (small) amount δ. It follows from the first part of Fact B that this change will *increase* the value of the objective function by some amount $\Delta > 0$.[4] Now, decrease the value of

[3]The proof of this fact depends on which case -1 through 6- we are considering. In each case, it is straightforward to identify a deviation that does not violate either constraint and improves the utility. For the sake of brevity, we do not include the relevant calculations.

[4]We know from Fact B that if we increase $\gamma\left(k'\right)$ and decrease $\gamma\left(k''\right)$, where $k' = k_n, ..., n-2$ and $k'' = k'+1, ..., n-1$, then the expected utility decreases. Notice that $\bar{k} \leqslant n-1$. Therefore, if we decrease the value of $\gamma\left(k\right)$ for some $k = k_n, \ldots, \bar{k}-1$ and increase the value of $\gamma\left(\bar{k}\right)$ (i.e., we take a "mirror image" of the type of changes described in Fact B), then the expected utility must increase.

$\gamma(k)$ and increase the value of $\gamma\left(\tilde{k}\right)$, for some $\tilde{k} = 1, \ldots, \hat{k}$, so that the LHS of both constraints increases by δ given above. This will *decrease* the value of the objective function by $\Delta' > 0$. The second part of Fact B implies that $\Delta > \Delta'$ and so the the combination of the two changes is feasible and strictly beneficial.

Proof of Fact A

The goal of this section is as follows. Fix $k' = k_n \left(= \frac{n+1}{2}\right), \ldots, n-1$ and $k = 1, \ldots, \frac{n+1}{2} - 2 \left(= \frac{n-3}{2}\right)$. Suppose that we decrease $\gamma(k')$ by $\eta > 0$ and increase the value of $\gamma(k)$ by $\varepsilon > 0$ to increase the LHS of both constraints by the same (small) number $\delta > 0$ (we will show that this is possible). Let $Z(k)$ denote the change of the value of the objective function. We show that $Z(k) < Z(k+1) < 0$.

Consider k. To find ε and η, we need to solve

$$a(k)\varepsilon - a(k')\eta = \delta,$$
$$b(k)\varepsilon - b(k')\eta = \delta.$$

The solution to this system is

$$\varepsilon = \frac{a(k')-b(k')}{b(k)a(k')-b(k')a(k)}\delta,$$
$$\eta = \frac{a(k)}{a(k')}\frac{a(k')-b(k')}{b(k)a(k')-b(k')a(k)}\delta - \frac{1}{a(k')}\delta.$$

Notice that $a(k') - b(k') < 0$ and $a(k') < 0$. Thus, to show that $\varepsilon > 0$ and $\eta > 0$, it is necessary and sufficient that

$$b(k)a(k') - b(k')a(k) < 0.$$

To simplify the notation we define:

$$a_1(k) = \binom{n-1}{k-1}\frac{1}{k}(n(1-p)-k)qP(I)p^{n-k-1}$$
$$a_2(k) = \binom{n-1}{k-1}\frac{1}{k}(k-np)(1-q)P(G)p^{k}$$

so that

$$a(k) = a_1(k)(1-p)^k + a_2(k)(1-p)^{n-k-1}.$$

Similarly, define

$$b_1(k) = \binom{n-1}{k-1}\frac{1}{k}(k-n(1-p))qP(I)p^{n-k}$$
$$b_2(k) = \binom{n-1}{k-1}\frac{1}{k}(np-k)(1-q)P(G)p^{k-1}$$

so that

$$b(k) = b_1(k)(1-p)^{k-1} + b_2(k)(1-p)^{n-k}$$

Notice that

$$b_1(k)\,a_1(k') = b_1(k')\,a_1(k)$$
$$b_2(k)\,a_2(k') = b_2(k')\,a_2(k)$$

and so

$$\begin{gathered}
b(k)\,a(k') - b(k')\,a(k) = \\
b_1(k)\,a_2(k')\,(1-p)^{n-k'+k-2} + b_2(k)\,a_1(k')\,(1-p)^{n-k+k'} \\
-b_1(k')\,a_2(k)\,(1-p)^{n-k+k'-2} - b_2(k')\,a_1(k)\,(1-p)^{n-k'+k}.
\end{gathered}$$

Note that the smallest power of the term $(1-p)$ in the expression above is $n-k'+k-2$. Therefore, for p close to 1 the sign of $b(k)\,a(k') - b(k')\,a(k)$ coincides with the sign of $b_1(k)\,a_2(k')$, which is negative.

The total effect $Z(k)$ on the utility is then

$$\begin{gathered}
Z(k) = v(k)\,\varepsilon - v(k')\,\eta = \\
\left(v(k) - v(k')\,\tfrac{a(k)}{a(k')}\right) \tfrac{a(k')-b(k')}{b(k)a(k')-b(k')a(k)}\delta + \tfrac{v(k')}{a(k')}\delta,
\end{gathered}$$

which is negative. In a similar way, for $k+1$ we get

$$Z(k+1) = \left(v(k+1) - v(k')\,\frac{a(k+1)}{a(k')}\right) \frac{a(k')-b(k')}{b(k+1)\,a(k') - b(k')\,a(k+1)}\delta + \frac{v(k')}{a(k')}\delta.$$

Recall that we need to show that $Z(k+1) > Z(k)$. We subtract $\frac{v(k')}{a(k')}\delta$ from $Z(k)$ and $Z(k+1)$. We then multiply both terms by the positive quantity (recall $a(k') < 0$ and $b(k') > 0$)

$$\frac{a(k')}{a(k') - b(k')}\frac{1}{\delta}.$$

We need to show

$$\frac{v(k+1)\,a(k') - v(k')\,a(k+1)}{b(k+1)\,a(k') - b(k')\,a(k+1)} > \frac{v(k)\,a(k') - v(k')\,a(k)}{b(k)\,a(k') - b(k')\,a(k)}.$$

We multiply both sides by

$$[b(k+1)\,a(k') - b(k')\,a(k+1)]\,[b(k)\,a(k') - b(k')\,a(k)] > 0$$

and obtain

$$\begin{gathered}
[v(k+1)\,a(k') - v(k')\,a(k+1)]\,[b(k)\,a(k') - b(k')\,a(k)] > \\
[v(k)\,a(k') - v(k')\,a(k)]\,[b(k+1)\,a(k') - b(k')\,a(k+1)].
\end{gathered} \tag{2}$$

Each side of the inequality contains several terms. However, as p approaches 1, it suffices to consider the terms with the smallest power of $(1-p)$ to determine whether the inequality is satisfied or not.

We now write

$$v(k) = v_1(k)(1-p)^k + v_2(k)(1-p)^{n-k},$$

where we define

$$v_1(k) = -\binom{n}{k} qP(I)\, p^{n-k},$$
$$v_2(k) = \binom{n}{k}(1-q)\, P(G)\, p^k.$$

Then,

$$v(k)a(k') - v(k')a(k) = v_1(k)a_1(k')(1-p)^{k+k'} + v_1(k)a_2(k')(1-p)^{k+n-k'-1} + v_2(k)a_1(k')(1-p)^{n-k+k'} + v_2(k)a_2(k')(1-p)^{2n-k-k'-1} - v_1(k')a_1(k)(1-p)^{k+k'} - v_1(k')a_2(k)(1-p)^{k'+n-k-1} - v_2(k')a_1(k)(1-p)^{n-k'+k} - v_2(k')a_2(k)(1-p)^{2n-k-k'-1}.$$

The smallest power of $(1-p)$ is $k+n-k'-1$ (similarly, if we switch k with $k+1$, the smallest power would be $k+n-k'$).

Consider now the LHS of inequality (2):

$$[v(k+1)a(k') - v(k')a(k+1)][b(k)a(k') - b(k')a(k)].$$

The term with the smallest power of $(1-p)$ is $v_1(k+1)a_2(k')b_1(k)a_2(k')$ and that power is $2(n-k'-1+k)$.

Consider the RHS of inequality (2):

$$[v(k)a(k') - v(k')a(k)][b(k+1)a(k') - b(k')a(k+1)].$$

The term with the smallest power of $(1-p)$ is $v_1(k)a_2(k')b_1(k+1)a_2(k')$ and that power is $2(n-k'-1+k)$.

Thus, the two sides have the same powers and we have to show that

$$v_1(k+1)b_1(k)(a_2(k'))^2 > v_1(k)b_1(k+1)(a_2(k'))^2.$$

We divide both sides by $(a_2(k'))^2$ and compute the value of

$$v_1(k+1)b_1(k) - v_1(k)b_1(k+1)$$

when $p=1$ (by continuity, the sign of the expression extends to p close to 1).

When $p=1$,

$$v_1(k+1)b_1(k) - v_1(k)b_1(k+1) =$$
$$(qP(I))^2\left[-\binom{n}{k+1}\binom{n-1}{k-1} + \binom{n}{k}\binom{n-1}{k}\right] =$$

$$(qP(I))^2\left[-\frac{n!}{(k+1)!(n-k-1)!}\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{n!}{k!(n-k)!}\frac{(n-1)!}{k!(n-k-1)!}\right]=$$
$$(qP(I))^2\frac{n!(n-1)!}{(n-k-1)!(n-k)!(k!)^2}\left(-\frac{k}{k+1}+1\right)>0.$$

This concludes the proof of Fact A.

Proof of Fact B

In this section we will prove the following. Consider $k'=k_n\left(=\frac{n+1}{2}\right),\ldots,n-2$, $k''=k'+1,\ldots,n-1$ and $k=1,\ldots,\frac{n-1}{2}$. Consider two different courses of action. In the first one, we decrease $\gamma(k')$ by $\eta>0$ and increase the value of $\gamma(k)$ by $\varepsilon>0$ to increase the LHS of both constraints by the same (small) number $\delta>0$. Let $Z(k)$ denote the corresponding change of the value of the objective function (this is the case analyzed in the previous section). In the second course of action, we increase $\gamma(k')$ by $\eta>0$ and decrease the value of $\gamma(k'')$ by $\varepsilon>0$ to increase the LHS of both constraints by the same (small) number $\delta>0$. This will change the value of the objective function by $\bar{Z}(k'')$. We want to show that $\bar{Z}(k'')<Z(k)$. (Recall that $Z(k)<0$. Thus, the inequality $\bar{Z}(k'')<Z(k)$ will also prove the first part of Fact B.)

Consider the second course of action. We need to solve the following system of equations:

$$-a(k'')\varepsilon+a(k')\eta=\delta,$$
$$-b(k'')\varepsilon+b(k')\eta=\delta.$$

The solution is

$$\varepsilon=\frac{a(k')-b(k')}{b(k')a(k'')-b(k'')a(k')}\delta,$$
$$\eta=\frac{a(k'')}{a(k')}\frac{a(k')-b(k')}{b(k')a(k'')-b(k'')a(k')}\delta+\frac{1}{a(k')}\delta.$$

It is simple to check that when p is close to 1 both ε and η are positive. Notice also that the denominator of ε is negative.

The total effect on the objective function $\bar{Z}(k'')$ is equal to

$$\bar{Z}(k'')=v(k')\eta-v(k'')\varepsilon=$$
$$\left(v(k')\frac{a(k'')}{a(k')}-v(k'')\right)\frac{a(k')-b(k')}{b(k')a(k'')-b(k'')a(k')}\delta+\frac{v(k')}{a(k')}\delta.$$

Recall that $Z(k)$ is equal to

$$Z(k)=\left(v(k)-v(k')\frac{a(k)}{a(k')}\right)\frac{a(k')-b(k')}{b(k)a(k')-b(k')a(k)}\delta+\frac{v(k')}{a(k')}\delta.$$

We subtract $\frac{v(k')}{a(k')}\delta$ from both $\bar{Z}(k'')$ and $Z(k)$ and multiply both by $\delta\frac{a(k')}{a(k')-b(k')}>0$. It remains to show that

$$\frac{v(k)a(k')-v(k')a(k)}{b(k)a(k')-b(k')a(k)}>\frac{v(k')a(k'')-v(k'')a(k')}{b(k')a(k'')-b(k'')a(k')}.$$

We multiply both sides by $[b(k)\, a(k') - b(k')\, a(k)]\, [b(k')\, a(k'') - b(k'')\, a(k')] > 0$ and get

$$\begin{aligned} & [v(k)\, a(k') - v(k')\, a(k)]\, [b(k')\, a(k'') - b(k'')\, a(k')] > \\ & [v(k')\, a(k'') - v(k'')\, a(k')]\, [b(k)\, a(k') - b(k')\, a(k)]\,. \end{aligned} \tag{3}$$

For each term inside the square brackets we now identify the element with the smallest power of $(1-p)$.

We already know from the previous section that for $[b(k)\, a(k') - b(k')\, a(k)]$ we select $b_1(k)\, a_2(k')\, (1-p)^{n-k'+k-2}$.

In a similar way, for $[b(k')\, a(k'') - b(k'')\, a(k')]$ we select $b_1(k')\, a_2(k'')\, (1-p)^{n-k''+k'-2}$.

Consider now $[v(k)\, a(k') - v(k')\, a(k)]$. We select $v_1(k)\, a_2(k')\, (1-p)^{k+n-k'-1}$.

Finally, consider $[v(k')\, a(k'') - v(k'')\, a(k')]$. We select

$$[v_2(k')\, a_2(k'') - v_2(k'')\, a_2(k')]\, (1-p)^{2n-k'-k''-1}.$$

Thus for p close to 1, inequality (3) is satisfied if and only if the following inequality is satisfied:

$$v_1(k)\, a_2(k')\, b_1(k')\, a_2(k'')\, (1-p)^{2n-k''+k-3} >$$

$$[v_2(k')\, a_2(k'') - v_2(k'')\, a_2(k')]\, b_1(k)\, a_2(k')\, (1-p)^{3n+k-2k'-k''-3}.$$

The exponent of the RHS is strictly smaller than the exponent of the LHS. Thus, it suffices to show

$$[v_2(k')\, a_2(k'') - v_2(k'')\, a_2(k')]\, b_1(k)\, a_2(k') < 0.$$

Notice that for p close to 1, $b_1(k)\, a_2(k') < 0$. We now evaluate the difference $v_2(k')\, a_2(k'') - v_2(k'')\, a_2(k')$ at $p = 1$ and show that it is positive. By continuity, the above inequality will be satisfied when p is close to 1.

When $p = 1$,

$$\begin{aligned} & v_2(k')\, a_2(k'') - v_2(k'')\, a_2(k') = \\ & ((1-q)\, P(G))^2 \left[\binom{n}{k'}\binom{n-1}{k''-1}\frac{k''-n}{k''} - \binom{n}{k''}\binom{n-1}{k'-1}\frac{k'-n}{k'}\right] = \\ & ((1-q)\, P(G))^2\, \frac{n!(n-1)!}{(n-k')!(n-k'')!k'!k''!}\, (k''-k') > 0. \end{aligned}$$

This concludes the proof of Fact B.

Proof of Fact C

Consider $k = 1, \ldots, k_n - 1 \left(= \frac{n-1}{2}\right)$ and $k' = k_n \left(= \frac{n+1}{2}\right), \ldots, n-1$. Suppose that we want to decrease the value of $\gamma(k)$ and increase the value of $\gamma(k')$ to increase the

LHS of both constraints by the same positive amount δ. We now show that this is impossible.

If the change described above is possible then there exist $\varepsilon > 0$ and $\eta > 0$ that solve the following system

$$-a(k)\varepsilon + a(k')\eta = \delta,$$
$$-b(k)\varepsilon + b(k')\eta = \delta.$$

The solution is

$$\varepsilon = \frac{a(k')-b(k')}{b(k')a(k)-b(k)a(k')}\delta,$$
$$\eta = \frac{a(k)}{a(k')}\frac{a(k')-b(k')}{b(k')a(k)-b(k)a(k')}\delta + \frac{1}{a(k')}\delta.$$

Notice that $a(k') - b(k') < 0$. Moreover, we know from the analysis above that for p close to 1 the sign of

$$b(k')a(k) - b(k)a(k')$$

coincides with the sign of $-b_1(k)a_2(k')$, which is positive. Thus, ε and η must be negative.

Proof of Fact D

Consider $k = k_n, \ldots, n-1$. Suppose that we want to increase both the value of $\gamma(k)$ and the value of $\gamma(0)$ to increase the LHS of both constraints by the same positive amount δ. We now show that this is impossible.

If the change described above is possible then there exist $\varepsilon > 0$ and $\eta > 0$ that solve the following system

$$a(0)\varepsilon + a(k)\eta = \delta,$$
$$b(0)\varepsilon + b(k)\eta = \delta.$$

The solution is

$$\varepsilon = \frac{a(k)-b(k)}{b(0)a(k)-b(k)a(0)}\delta,$$
$$\eta = -\frac{a(0)}{a(k)}\frac{a(k)-b(k)}{b(0)a(k)-b(k)a(0)}\delta + \frac{1}{a(k)}\delta.$$

Notice that $a(k) - b(k) < 0$. We now show that $b(0)a(k) - b(k)a(0)$ is positive, which implies that ε is negative.

Recall that

$$a(0) = -f(1;n) = qP(I)p^{n-1}(1-p) - (1-q)P(G)p(1-p)^{n-1}$$

and that

$$b(0) = f(0;n) = -qP(I)p^n + (1-q)P(G)(1-p)^n.$$

For p close to 1 the sign of $b(0)a(k) - b(k)a(0)$ coincides with the sign of $-qP(I)a_2(k)$ which is positive.

Proof of Fact E

Consider $k = k_n, \ldots, n-1$. Suppose that we want to decrease both the value of $\gamma(k)$ and the value of $\gamma(n)$ to increase the LHS of both constraints by the same positive amount δ. We now show that this is impossible.

If the change described above is possible then there exist $\varepsilon > 0$ and $\eta > 0$ that solve the following system

$$
\begin{aligned}
-a(k)\,\varepsilon - a(n)\,\eta &= \delta, \\
-b(k)\,\varepsilon - b(n)\,\eta &= \delta.
\end{aligned}
$$

The solution is

$$
\varepsilon = \tfrac{a(n)-b(n)}{b(n)a(k)-b(k)a(n)}\delta,
$$

$$
\eta = -\tfrac{a(k)}{a(n)}\tfrac{a(n)-b(n)}{b(n)a(k)-b(k)a(n)}\delta - \tfrac{1}{a(n)}\delta.
$$

Recall that

$$
a(n) = f(n;n) = -qP(I)(1-p)^n + (1-q)P(G)p^n
$$

and that

$$
b(n) = -f(n-1;n) = qP(I)\,p\,(1-p)^{n-1} - (1-q)P(G)p^{n-1}(1-p).
$$

Define $a_1(n) = -qP(I)$ and $a_2(n) = (1-q)P(G)p^n$. Also, define $b_1(n) = qP(I)\,p$ and $b_2(n) = -(1-q)P(G)p^{n-1}$.

The numerator of ε is positive. We now show that the denominator of ε is negative.

We have to show $b(n)a(k) - b(k)a(n) < 0$ for p large. Notice that (after some simplifications)

$$
\begin{aligned}
b(n)a(k) - b(k)a(n) = {} & b_1(n)a_2(k)(1-p)^{2n-k-2} + b_2(n)a_1(k)(1-p)^{k+1} \\
& - b_1(k)a_2(n)(1-p)^{k-1} - b_2(k)a_1(n)(1-p)^{2n-k}.
\end{aligned}
$$

The smallest power of $(1-p)$ is $k-1$, and thus for p close to 1 the sign of $b(n)a(k) - b(k)a(n)$ coincides with the sign of $-b_1(k)a_2(n)$ which is negative.

Remark 3 *Suppose that there exists a cost c' such that the optimal device takes the form*

$$
\begin{aligned}
&\gamma(0) = 0, \quad \gamma(1) = \ldots = \gamma(k_n - 1) = 1, \quad \gamma(k_n) = \ldots = \gamma(k'-1) = 0 \\
&\gamma(k') = \alpha \quad \gamma(k'+1) = \ldots = \gamma(n) = 1
\end{aligned}
\tag{4}
$$

then $k' = n-1$ and $\alpha < 1$.

Similarly, suppose that there exists a cost c'' such that the optimal device takes the form

$$\bar{\gamma}_n(0) = \ldots = \bar{\gamma}_n(k''-1) = 0, \quad \bar{\gamma}_n(k'') = \beta, \quad \bar{\gamma}_n(k''+1) = \ldots = \bar{\gamma}_n(k_n - 1) = 1,$$
$$\bar{\gamma}_n(k_n) = \ldots = \bar{\gamma}_n(n-1) = 0, \quad \bar{\gamma}_n(n) = 1,$$

then $k'' = 1$ and $\beta > 0$.

An implication of the first part of the remark is the following. Suppose k' were smaller than $n-1$, and consider a cost c above c'. To satisfy the constraints, we could increase the value of $\gamma(k')$ and decrease the value of $\gamma(k)$ for some $k = k'+1, \ldots, n-1$. On the other hand, if $k' = n-1$ as claimed then it is impossible to modify the mechanism in order to satisfy both constraints. A similar implication follows from the second part of the remark and therefore the optimal device must take the form specified in Proposition 3.

Proof of Remark 3

We provide the proof for the first claim. The proof for the second claim is analogous.

To see that $k' = n-1$ when p is close to 1, consider the device described in (4). Both constraints are satisfied with equality. Thus,

$$f(1;n) - \binom{n-1}{\frac{n-1}{2}} f\left(\tfrac{n+1}{2}\right) + \alpha \binom{n-1}{k'-1} f(k';n) + (1-\alpha)\binom{n-1}{k'} f(k'+1;n) =$$
$$-f(0;n) + \binom{n-1}{\frac{n-1}{2}} f\left(\tfrac{n-1}{2}\right) - \alpha \binom{n-1}{k'-1} f(k'-1;n) - (1-\alpha)\binom{n-1}{k'} f(k';n)$$

(and both sides are equal to c'). Notice that as p approaches 1 the RHS of the equality converges to $qP(I)$ (since $-f(0;n)$ contains the term $qP(I)p^n$ and every other term contains $(1-p)^r$ for some $r > 0$). If $k' < n-1$, the LHS converges to zero (since each term contains $(1-p)^r$ for some $r > 0$) and the equality cannot be satisfied.

2 Distortionary Mechanisms when N is Fixed and p is Close to 1

In Proposition 2 we fix q, $P(I)$, p and let N go to infinity. In Proposition 3 and the notes above, we fix N and let p approach 1. The following Proposition extends Proposition 2 and provides conditions for the optimal extended mechanism to involve distortions when, indeed, N is fixed and p is large.

Proposition 2^* *Fix N, q and $P(I)$ and assume that either $qP(I) > 2(1-q)P(G)$ or $qP(I) < \frac{1}{2}(1-q)P(G)$. There exists $\tilde{p} < 1$ such that for every $p > \tilde{p}$ the following holds. For any $n = 2, ..., N$, suppose that the Bayesian device with n agents is admissible. Then there exists an admissible distortionary device with $n+1$ agents that yields greater expected utility than $\hat{V}(n)$.*

Proof of Proposition 2*

To simplify the notation, we define $D \equiv qP(I)$ and $E \equiv (1-q)P(G)$. The proof depends on which of the two cases specified in the proposition holds and on whether n is even or odd. We present the proof for the case $D > 2E$ and n odd (so that $n \geqslant 3$). The other three cases follow analogously.

When p is close to 1, and n is odd, then $k_n = \frac{n+1}{2}$. Moreover, $z(n)$ is strictly larger than $\frac{n}{2}$ but very close to $\frac{n}{2}$. In particular, $k_n - z(n) < \frac{1}{2}$.

We now adapt the proof of Proposition 2. Clearly, when p is close to 1, the inequalities used in the proof of Proposition 2: $k_n - 1 \geqslant n(1-p)$ and $k_n \leqslant np$, are satisfied. As in the proof of Proposition 2 we need to show that $\alpha_2 < \alpha^*$ and $\alpha_2 < \alpha_1$, where

$$\alpha_1 = \frac{\binom{n}{k_n} f(k_n+1; n+1) - \binom{n-1}{k_n-1} f(k_n; n)}{\binom{n}{k_n} f(k_n+1; n+1) - \binom{n}{k_n-1} f(k_n; n+1)},$$

$$\alpha_2 = \frac{\binom{n-1}{k_n-1} f(k_n; n) + \binom{n}{k_n} f(k_n; n+1)}{\binom{n}{k_n} f(k_n; n+1) - \binom{n}{k_n-1} f(k_n-1; n+1)},$$

and

$$\alpha^* = \frac{n-k_n+1}{n+1}.$$

The denominators of α_1 and α_2 are positive. We begin with the inequality $\alpha^* > \alpha_2$. We need to show

$$\left(n - \tfrac{n+1}{2} + 1\right)\left[\binom{n}{\frac{n+1}{2}} f\left(\tfrac{n+1}{2}; n+1\right) - \binom{n}{\frac{n+1}{2}-1} f\left(\tfrac{n+1}{2} - 1; n+1\right)\right] >$$
$$(n+1)\left[\binom{n-1}{\frac{n+1}{2}-1} f\left(\tfrac{n+1}{2}; n\right) + \binom{n}{\frac{n+1}{2}} f\left(\tfrac{n+1}{2}; n+1\right)\right].$$

The easiest way to show that the inequality is satisfied for p close to 1 is to identify, for each term $f(k'; n')$, the component with the smallest power of $(1-p)$.

For $f\left(\frac{n+1}{2}; n+1\right)$ we select $-D(1-p)^{\frac{n+1}{2}} p^{\frac{n+1}{2}} + E(1-p)^{\frac{n+1}{2}} p^{\frac{n+1}{2}}$.

For $f\left(\frac{n-1}{2}; n+1\right)$ we select $-D(1-p)^{\frac{n-1}{2}} p^{\frac{n+3}{2}}$.

For $f\left(\frac{n+1}{2}; n\right)$ we select $E(1-p)^{\frac{n-1}{2}} p^{\frac{n+1}{2}}$.

Thus, when p is sufficiently close to 1, the above inequality is satisfied if and only if

$$\frac{n+1}{2}\binom{n}{\frac{n-1}{2}} D > (n+1)\binom{n-1}{\frac{n-1}{2}} E$$

which is equivalent to

$$\frac{n}{n+1} D > E.$$

Clearly, if $D > 2E$ then the inequality is satisfied for every $n \geqslant 3$.

Consider now the inequality $\alpha_1 > \alpha_2$. We need to show (recall the denominators are positive):

$$\left[\binom{n}{\frac{n+1}{2}} f\left(\tfrac{n+3}{2}; n+1\right) - \binom{n-1}{\frac{n-1}{2}} f\left(\tfrac{n+1}{2}; n\right)\right] \left[\binom{n}{\frac{n+1}{2}} f\left(\tfrac{n+1}{2}; n+1\right) - \binom{n}{\frac{n-1}{2}} f\left(\tfrac{n-1}{2}; n+1\right)\right] >$$
$$\left[\binom{n-1}{\frac{n-1}{2}} f\left(\tfrac{n+1}{2}; n\right) + \binom{n}{\frac{n+1}{2}} f\left(\tfrac{n+1}{2}; n+1\right)\right] \left[\binom{n}{\frac{n+1}{2}} f\left(\tfrac{n+3}{2}; n+1\right) - \binom{n}{\frac{n-1}{2}} f\left(\tfrac{n+1}{2}; n+1\right)\right]$$

We proceed as above and identify the components with the smallest power of $(1-p)$.

For $f\left(\frac{n+3}{2}; n+1\right)$ we select $E\,(1-p)^{\frac{n-1}{2}}\, p^{\frac{n+3}{2}}$.

For $f\left(\frac{n+1}{2}; n+1\right)$ we select $-D\,(1-p)^{\frac{n+1}{2}}\, p^{\frac{n+1}{2}} + E\,(1-p)^{\frac{n+1}{2}}\, p^{\frac{n+1}{2}}$.

For $f\left(\frac{n-1}{2}; n+1\right)$ we select $-D\,(1-p)^{\frac{n-1}{2}}\, p^{\frac{n+3}{2}}$.

For $f\left(\frac{n+1}{2}; n\right)$ we select $E\,(1-p)^{\frac{n-1}{2}}\, p^{\frac{n+1}{2}}$.

Thus, we need to show

$$E\left[\binom{n}{\frac{n+1}{2}} - \binom{n-1}{\frac{n-1}{2}}\right](1-p)^{\frac{n-1}{2}}\, D\binom{n}{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}} >$$
$$E\binom{n-1}{\frac{n-1}{2}}(1-p)^{\frac{n-1}{2}}\, E\binom{n}{\frac{n+1}{2}}(1-p)^{\frac{n-1}{2}}.$$

We divide both sides by $E\,(1-p)^{n-1}$ yielding

$$D\left[\binom{n}{\frac{n+1}{2}} - \binom{n-1}{\frac{n-1}{2}}\right]\binom{n}{\frac{n-1}{2}} > E\binom{n-1}{\frac{n-1}{2}}\binom{n}{\frac{n+1}{2}}$$

Notice that $\binom{n}{\frac{n+1}{2}} = \frac{n}{\frac{n+1}{2}}\binom{n-1}{\frac{n-1}{2}}$ and that $\binom{n}{\frac{n+1}{2}} = \binom{n}{\frac{n-1}{2}}$. Therefore, the inequality above translates into

$$D\binom{n-1}{\frac{n-1}{2}}\left[\frac{n}{\frac{n+1}{2}} - 1\right]\binom{n}{\frac{n-1}{2}} > E\binom{n-1}{\frac{n-1}{2}}\binom{n}{\frac{n+1}{2}}.$$

We divide both sides by $\binom{n-1}{\frac{n-1}{2}}\binom{n}{\frac{n-1}{2}}$ and get

$$D\left(\frac{2n}{n+1} - 1\right) > E$$

The inequality is satisfied for every odd $n \geqslant 3$ provided that $D > 2E$.

References

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